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# Relativistic Spherical Functions

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The finite-dimensional representations of the rotation groups in four-dimensional pseudo-Euclidian and Euclidean spaces are considered. The basis functions which are constructed are eigenfunctions of the four-dimensional angular momentum operator. The Clebsch-Gordon coefficients are calculated. Spinors which are eigenfunctions of the four-dimensional angular momentum operator are determined. The possibility of separating variables in the relativistic two-body problem is discussed briefly.

## 1. STATEMENT OF THE PROBLEM

FOR the nonrelativistic description of the bound state of a two-particle system we choose as wave functions the basis functions of the irreducible representations of the three-dimensional rotation group. These functions are also the eigenfunctions of the angular momentum operator. By analogy with the three-dimensional case, we can construct a system of eigenfunctions of the four-dimensional angular momenta operator, which form the basis for irreducible representations of the Lorentz group, and try to use them for solving the equation of a system of relativistic particles.

If the relativistic equation admits of analytic continuation into the region of pure imaginary values of the relative time, its solution is greatly simplified. In this case one can use the basis functions of the rotation group in Euclidean space. The possibility of analytic continuation for the relativistic two-body equation (equation of the Bethe-Salpeter type) was pointed out by Wick[1]. The four-dimensional spherical functions of a Euclidean space are well known, and were used by Fock[2] for the solution of the problem of the hydrogen atom in momentum representation. They have the form

$$\psi_{nlm}(\alpha, \vartheta, \varphi) = i^l Y_{lm}(\vartheta\varphi) \frac{\sin^l \alpha}{M_l} \frac{d^{l+1} \cos n\alpha}{d(\cos\alpha)^{l+1}}, \quad (1)$$

where $\alpha$, $\vartheta$, $\varphi$ are the angles describing the four-dimensional radius vector, and

$$M_l = [n^2(n^2 - 1^2) \ldots (n^2 - l^2)]^{1/2}, \quad n = 0, 1, 2, \ldots$$

These functions are eigenfunctions of the square of the four-dimensional angular momentum operator $L^2$ ($L^2$ is the angular part of the d'Alembertian operator) and can be used as basis functions.

The irreducible finite-dimensional representations of the group can be obtained from the irreducible representations of the group of rotations of four-dimensional Euclidean space by going over to pure imaginary values of the angle $\alpha$. If in Eq. (1) we simultaneously replace $\alpha$ by $i\alpha$ and $n$ by $in$, where $0 \leq n \leq \infty$, we get one of the infinite-dimensional representations of the Lorentz group. The infinite-dimensional representations were investigated in general form by Gel'fand and Naimark[3]. A detailed survey of the linear representations of the Lorentz group is contained in a paper of Naimark[4]. According to this work, the representations can be classified by the assignment of two numbers $k_0$, $c$. The number $k_0$ determines the smallest weight of representation of the three-dimensional subgroup which is contained in the $(k_0, c)$ representation of the Lorentz group. The explicit form of the basis functions for one special case with $k_0 \neq 0$ was obtained in a paper of Ginzburg and Tamm[5].

Since we are interested in representations which contain a scalar among the basis functions, we shall consider the case of $k_0 = 0$.

In order to make use of the invariance of equations describing a system of relativistic particles with respect to the choice of the reference system (i.e., with respect to Lorentz transformations), we must know the explicit form of the expansion of a wave function in terms of irreducible representations (Clebsch-Gordan series). The Clebsch-Gordan coefficients for the three-dimensional case are well known[6]. They are obtained for the four-dimensional case in the present paper. If the basis functions are chosen in the form of Eq. (1), then the expansion turns out to be extremely complicated. However, we can make use of the fact that $L^2$ and $l^2$ commute with one another, and take our basis functions to be linear combinations of the $\psi_{nlm}$. Linear combinations for which the Clebsch-Gordan expansion is particularly simple are obtained in a natural way, if we construct the basis functions starting from the spinor representations of the four-dimensional rotation group. These new basis functions prove to be suited for the description of systems of relativistic particles.

## 2. THE MATRICES OF THE IRREDUCIBLE REPRESENTATIONS OF THE FOUR-DIMENSIONAL ROTATION GROUP

We first construct the finite-dimensional, irreducible representations of the Lorentz group and determine the coefficients in the Clebsch-Gordan expansion. A special feature of this case is the need to consider timelike and spacelike functions separately. If we replace $\alpha$ by $i\alpha$, both of them give equivalent representations of the Euclidean group of four-dimensional rotations.

We introduce, in the four-dimensional pseudo-Euclidean space, the coordinates

$$t = \rho \cosh \alpha, \; r = \rho \sinh \alpha \text{ for } t^2 - r^2 > 0, \tag{2}$$

$$t = \rho \sinh \alpha, \; r = \rho \cosh \alpha \text{ for } r^2 - t^2 > 0. \tag{3}$$

The Lorentz transformation is a rotation of the four-dimensional coordinate system in the pseudo-Euclidean space. Let $\epsilon$, $\zeta$ and $\epsilon'$, $\zeta'$ be the two components of a spinor in the initial and rotated coordinate systems, respectively. We denote the complex-conjugate spinors by $\dot\epsilon$, $\dot\zeta$, $\dot\epsilon'$, $\dot\zeta'$. In Ref. 7 it is shown that the $(2J+1)(2j+1)$ quantities

$$u^{Jj}_{M\mu} = [(J+M)!\,(J-M)!]^{-1/2}\epsilon^{J+M}\zeta^{J-M} \times [(j+\mu)!\,(j-\mu)!]^{-1/2}\dot\epsilon^{j+\mu}\dot\zeta^{j-\mu} \tag{4}$$

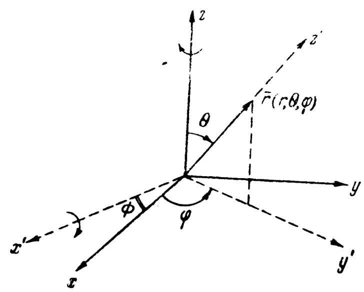
transform according to an irreducible representation of the Lorentz group. We apply to $u^{Jj}_{M\mu}(\alpha', \vartheta', \varphi')$, where $\alpha'$, $\vartheta'$, $\varphi'$ are the "polar angles" in the new system, a four-dimensional rotation operator $\hat R$, which rotates the system back to its initial position

$$\hat R u^{Jj}_{M\mu}(\alpha', \vartheta', \psi') \equiv u^{Jj}_{M\mu}(\alpha, \vartheta, \varphi) = \sum_{M',\mu'} D^{Jj}_{M\mu M'\mu'}(\psi, \theta, \phi)\, u^{Jj}_{M'\mu'}(\alpha', \vartheta', \varphi'). \tag{5}$$

The coefficients $D$ are the components of the $(2J+1)(2j+1)$-dimensional matrix of the transformation, $J > M > -J$; $j > \mu > -j$; $\theta$ and $\phi$ are the Euler angles of the three-dimensional rotation (cf. Fig. 1). The rotation about the new position of the $z$ axis (through the third Euler angle ) is not of interest, since a point in space is determined by a pair of angles. Tanh $\psi = v$ is the velocity of the new reference system. The $z$ axis is chosen along the direction of the velocity.

Under a four-dimensional rotation of the reference system, spinors transform according to the law $\epsilon = \delta\epsilon' - \beta\zeta'$, $\zeta = -\gamma\epsilon' + \alpha\zeta'$. The quantities $\alpha$, $\beta$, $\gamma$, $\delta$ form a binary transformation matrix:

$$\alpha = \cos\frac{\theta}{2}\exp\{(i\phi - \psi)/2\}; \quad \beta = -i\sin\frac{\theta}{2}\exp\{-(i\phi+\psi)/2\}; \quad \gamma = -i\sin\frac{\theta}{2}\exp\{(i\phi+\psi)/2\}; \quad \delta = \cos\frac{\theta}{2}\exp\{-(i\phi-\psi)/2\}. \tag{6}$$




Upon substituting the explicit expressions (4) and (6) in (5), we get

$$D^{Jj}_{M\mu M'\mu'}\left(\psi, \theta, \varphi - \frac{\pi}{2}\right) = D^{J*}_{M'M}\left(-\frac{\pi}{2}, \theta, \varphi\right) D^{j}_{\mu'\mu}\left(-\frac{\pi}{2}, \theta, \varphi\right) e^{(M'+\mu')\psi}, \tag{7}$$

where $D^J_{MM'}(-\pi/2, \theta, \phi)$ and $D^j_{\mu\mu'}(-\pi/2, \theta, \phi)$ are the matrices of the irreducible $(2J+1)$ and $(2j+1)$-dimensional representations of the three-dimensional rotation group:

$$D^J_{M'M}(\gamma, \theta, \varphi) = \sum_k (-1)^k \frac{[(J+M)!\,(J-M)!\,(J+M')!\,(J-M')!]^{1/2}}{(J+M-k)!\,k!\,(M'-M+k)!\,(J-M'-k)!} \times e^{iM'\gamma}\left[\cos\frac{\theta}{2}\right]^{2J+M-M'-2k}\left[\sin\frac{\theta}{2}\right]^{2k-M-M'} e^{iM\varphi}. \tag{8}$$

Making use of the equality

$$D^{J*}_{M'M}(\gamma, \theta, \varphi) = (-1)^{M-M'} D^J_{-M'-M}(\gamma, \theta, \varphi) \tag{9}$$

and the Clebsch-Gordan expansion[6]

$$D^J_{-M'-M} D^j_{\mu'\mu} = \sum_{l=|J-j|}^{l=J+j} C^{lm'}_{J-M'j\mu'} C^{lm}_{J-Mj\mu} D^l_{m'm}, \tag{10}$$

we obtain

$$D^{Jj}_{M\mu M'\mu'}\left(\psi, \theta, \varphi - \frac{\pi}{2}\right) = (-1)^{M-M'} e^{(M'+\mu')\psi} \times \sum_l C^{lm'}_{J-M'j\mu'} C^{lm}_{J-Mj\mu} D^l_{m'm}\left(-\frac{\pi}{2}, \theta, \varphi\right). \tag{11}$$

By using (10) and (11) we can get the Clebsch-Gordan expansion for the matrices in the four-dimensional space

$$D^{Jj}_{M\nu\alpha\beta} D^{Ll}_{\Lambda\lambda\gamma\delta} = \sum_{K=|J-L|}^{J+L} \sum_{N=|j-l|}^{j+l} C^{K\varkappa}_{JML\Lambda} C^{N\nu}_{j\mu l\lambda} C^{K\sigma}_{J\alpha L\gamma} C^{N\omega}_{j\beta l\delta} D^{KN}_{\varkappa\nu\sigma\omega}. \tag{12}$$

The orthogonality relations

$$\sum_{\mu_1} C^{j\mu}_{j_1\mu_1 j_2\mu-\mu_1} C^{j'\mu}_{j_1\mu_1 j_2\mu-\mu_1} = \delta_{jj'} \tag{13}$$

enable us to obtain the expansion which is reciprocal to (12):

$$D^{KN}_{\varkappa\nu\sigma\omega} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ M,\Lambda,\mu,\lambda}} C^{K\varkappa}_{JML\Lambda} C^{N\nu}_{j\mu l\lambda} C^{K\sigma}_{J\alpha L\gamma} C^{N\omega}_{j\beta l\delta} D^{Jj}_{M\mu\alpha\beta} D^{Ll}_{\Lambda\lambda\gamma\delta}. \tag{14}$$

If the rotation $\psi, \theta, \phi$ can be represented as two successive rotations $\psi'', \theta'', \phi''$ and $\psi', \theta', \phi'$, then

$$D^{Jj}_{M\mu M'\mu'}(\psi, \theta, \phi) = \sum_{\varkappa,\varkappa'} D^{Jj}_{M\mu\varkappa\varkappa'}(\psi', \theta', \phi') D^{Jj}_{\varkappa\varkappa' N'\mu'}(\psi'', \theta'', \phi''). \tag{15}$$

### 3. CONSTRUCTION OF RELATIVISTIC SPHERICAL FUNCTIONS

In the four-dimensional pseudo-Euclidean space, we select an aggregate of functions $Z^{Jj}_{M\mu}(\alpha, \vartheta, \varphi)$ which transform according to the $(2J+1)(2j+1)$-dimensional representation of the Lorentz group, and for which $Z^{Jj}_{M\mu}(0, 0, \pi/2) = \delta_{M\mu}$. According to Eq. (11), such a choice is possible. In view of (5), we obtain

$$Z^{Jj}_{M\mu}(\alpha, \vartheta, \varphi) = \sum_k D^{Jj}_{M\mu kk}\left(\alpha, \vartheta, \varphi - \frac{\pi}{2}\right). \tag{16}$$

If we substitute in (16) the expansion (11) and the equality

$$Y_{lm}(\vartheta\varphi) = \sqrt{\frac{2l+1}{4\pi}} (-1)^m D^l_{om}(\gamma, \vartheta, \varphi), \tag{17}$$

where $Y_{lm}(\vartheta, \varphi)$ is the three-dimensional spherical function [$Y_{lm}$ is here defined as in Ref. 8, and differs by a factor $(-1)^m$ from the definition of $Y_{lm}$ in, say, Ref. 9], we get:

$$Z^{Jj}_{M\mu}(\alpha, \vartheta, \varphi) = \sum_{lk} (-1)^{\mu-k} \sqrt{\frac{4\pi}{2l+1}} C^{lm}_{J-Mj\mu} Y_{lm}(\vartheta\varphi) C^{l0}_{J-kjk} e^{2k\alpha}. \tag{18}$$

It is known that the representations of the four-dimensional rotation group are determined by a pair of invariants, which can be constructed from the components of the infinitesimal rotation operator. If **M** is the space, and **N** the time part of the four-dimensional angular momentum operator $J$, then $J^2 = M^2 + N^2$ and **M**·**N** will be invariants. We consider a $Z$-function which depends only on the coordinates of the point and not on the spin variables. The operators **M** and **N** which act on it do not contain spin matrices and can be represented in the form

$$\mathbf{M} \equiv \mathbf{l} = -i[\mathbf{n}\nabla^\omega],$$

$$\mathbf{N} \equiv \mathbf{g} = \mathbf{n}\left(t\frac{\partial}{\partial r} + r\frac{\partial}{\partial t}\right) + \frac{t}{r}\nabla^\omega,$$

where **n** is a unit vector along $\mathbf{r} = \mathbf{n}r$, and $\nabla^\omega$ is the angular part of the gradient. Obviously $\mathbf{g}\cdot\mathbf{l} = 0$. In this case, the group representation is determined (in the notation of Ref. 4), by assigning the pair of numbers $k_0 = |J - j| = 0$, $c = 2J + 1$. The functions $Z^{JJ}_{M\mu} \equiv Z^J_{M\mu}$ transforms according to an irreducible representation of the Lorentz group. It is an eigenfunction of $L^2 = g^2 + l^2$, and can be chosen as a basis function. In the special case of $J = ½$ we have

$$Z^{1/2}_{1/2\,1/2} = \cosh\alpha + \sinh\alpha\cos\vartheta;$$
$$Z^{1/2}_{1/2\,-1/2} = \sinh\alpha\sin\vartheta\, e^{-i\varphi};$$
$$Z^{1/2}_{-1/2\,1/2} = \sinh\alpha\sin\vartheta\, e^{i\varphi};$$
$$Z^{1/2}_{-1/2\,-1/2} = \cosh\alpha - \sinh\alpha\cos\vartheta, \tag{19}$$

i.e., the components $Z^{1/2}_{M\mu}$ form a timelike four-dimensional vector with components:

$$t+z=\rho Z^{1/2}_{1/2\,1/2};\qquad t-z=\rho Z^{1/2}_{-1/2\,-1/2};\qquad x+iy=\rho Z^{1/2}_{-1/2\,1/2};\qquad x-iy=\rho Z^{1/2}_{1/2\,-1/2}. \tag{20}$$

The definition of the rotation angles in (6), so that $\tanh\psi = v$, limits us to the case where $t^2 - r^2 > 0$. Therefore the $Z^{Jj}_{M\mu}$ function defined by (18) may be called a timelike Z-function.

The connection between the spherical functions of the form (1) and the Z-functions can be established with the help of the relation

$$\psi_{nlm}=\sqrt{\frac{2J+1}{2l+1}}\sum_k(-1)^{l-J-k}C^{l0}_{J-kJk}e^{2k\alpha} = \frac{\sinh^l\alpha}{M_l}\frac{d^{l+1}\cosh n\alpha}{d(\cosh\alpha)^{l+1}}, \tag{21}$$

where $n = 2J + 1$. The validity of (21) can be checked by expanding the right side in series and comparing with the expansion of the left side. By using (21), we put $Z^J_{M\mu}$ in the form

$$Z^J_{M\mu}=\frac{\sqrt{4\pi}}{2J+1}\sum_l\sqrt{2l+1}\,C^{JM}_{J\mu l-m}\psi_{nlm}. \tag{22}$$

If in (21) we replace $\alpha$ by $i\alpha$, then the right side coincides with (1), so the expression (22) will determine the Z-function for a Euclidean space.

If we rotate through $\pi/2$ in the $(z,t)$ plane of the Euclidean space, a vector which became timelike when we replaced $\alpha$ by $-i\alpha$, will become spacelike. This enables us to construct spacelike basis functions of a finite-dimensional representation of the Lorentz group. Again, they will have the form (22), but now we must choose for $\psi_{nlm}$, not (21), but rather

$$\sqrt{\frac{2J+1}{2l+1}}\sum_k(-1)^{J-l}C^{l0}_{J-kJk}e^{2k\alpha} = i\,\frac{\cosh^l\alpha}{M_l}\frac{d^{l+1}\cos n\left(i\alpha+\frac{\pi}{2}\right)}{d(\sinh\alpha)^{l+1}} \tag{23}$$

The factor $(-1)^J$ has been inserted in (23) so that the Z-function will be real for arbitrary $J$. In analogy to (19), the components of the spacelike $Z^{1/2}_{M\mu}$ form a spacelike four-vector. The relations (2) remain valid, but we must use (3) for $x$, $y$, $z$, $t$ instead of (2). According to (12) and (16), the expansion

$$Z^J_{M\mu}Z^L_{\Lambda\lambda}=\sum_{N=|J-L|}^{J+L}C^{NM+\Lambda}_{JML\Lambda}C^{N\mu+\lambda}_{J\mu L\lambda}Z^N_{M+\Lambda,\,\mu+\lambda}; \tag{24}$$

$$Z^N_{M+\Lambda,\,\mu+\lambda}=\sum_{M,\Lambda,\mu,\lambda}C^{NM+\Lambda}_{JML\Lambda}C^{N\mu+\lambda}_{J\mu L\lambda}Z^J_{M\mu}Z^L_{\Lambda\lambda}. \tag{25}$$

exists for timelike Z-functions. The expansion (24) is valid for spacelike Z-functions, except that we must add a factor $(-1)^{N-J-L}$ on the right. This factor appears because we have introduced $(-1)^J$ in defining the functions.

By using (22) and (24), we obtain

$$\psi_{n_1l_1m_1}\psi_{n_2l_2m_2}=\sum C^{JM}_{J_1M_1J_2M_2}C^{J\mu}_{J_1\mu_1J_2\mu_2}C^{lm}_{J-MJ\mu}C^{l_1m_1}_{J_1-M_1J_1\mu_1}C^{l_2m_2}_{J_2-M_2J_2\mu_2}\psi_{nlm}; \quad n_i=2J_i+1. \tag{26}$$

The summation runs over $J$, $l$, $M_1$, $M_2$, $\mu_1$, $\mu_2$; it can be partially carried out using the Wigner 9-$j$ symbols:

$$\psi_{n_1l_1m_1}\psi_{n_2l_2m_2}=\sum_{J,l}(-1)^{J-J_1-J_2}C^{lm}_{l_1m_1l_2m_2}\,\chi(J_1J_1J_2J_2;\ l_1l_2,JJ,J)\,\psi_{nlm}, \tag{27}$$

where $\chi$ is related[10] to the 9–$j$ symbols and is a complicated function of $J_1$, $J_2$, $l_1$, $l_2$, $J$.

Comparison of (24) and (26), (27) shows that for problems where we have to use the expansion in irreducible representations, it is more convenient to choose the $Z^J_{M\mu}$ as basis functions, and not $\psi_{nlm}$.

Differentiation of the Z-function can be carried out, using (22) and the explicit form of $\psi_{nlm}$. However, this does not enable us directly to get the result in the form of an expansion in terms of irreducible representations (i.e., in terms of Z-functions). Such an expansion is frequently needed, and can be gotten as follows. We shall agree to denote the covariant components of the gradient by the symbol $\partial_{\alpha\beta}$, where $\alpha$ and $\beta$ refer to the component $Z^{1/2}_{\alpha\beta}$ which transforms like the associated differential operator

$$\partial_{\pm1/2,\,\pm1/2}=\partial/\partial t\mp\nabla_0;\qquad \partial_{\mp1/2,\,\pm1/2}=\mp\sqrt{2}\,\nabla_{\pm1}, \tag{28}$$

where $\nabla_0$, $\nabla_1$, $\nabla_{-1}$ are the cyclic components of $\nabla$. The cyclic components of a vector $\mathbf{a}$ are related to its Cartesian components by the relations $a_0 = a_z$,

$a_{\pm 1} = \pm 2^{-1/2}(a_x \pm i a_y)$. The contravariant components of the gradient are $\partial^{\alpha\beta} = (-1)^{\alpha+\beta+1} \times \partial_{-\alpha-\beta}$.

Making use of the transformation law for $\partial_{\alpha\beta}$ and the equality (24), we get

$$\partial_{\alpha\beta} G_J(\rho) Z^J_{M\mu} = \sum_l A_{Jl}(\rho) C^{lM+\alpha}_{1/2\alpha JM} C^{l\mu+\beta}_{1/2\beta J\mu} Z^l_{M+\alpha,\mu+\beta}, \quad (29)$$

where $G_J(\rho)$ depends only on $\rho$, and $A_{Jl}(\rho)$ is a quantity which can be found by starting from the explicit form of the operator $\partial_{\alpha\beta}$.

If $t^2 - r^2 > 0$, then

$$\partial_{\pm 1/2\, \pm 1/2} = Z^{1/2}_{\pm 1/2\, \pm 1/2} \frac{\partial}{\partial\rho} - \frac{1}{\rho}[\sinh\alpha \pm \cosh\alpha \cos\vartheta] \frac{\partial}{\partial\alpha} \mp \frac{\nabla_0^{\omega}}{\rho \sinh\alpha}; \quad (30)$$

$$\partial_{-1/2\, 1/2} = \partial^*_{1/2\, -1/2} = Z^{1/2}_{-1/2\, 1/2} \frac{\partial}{\partial\rho} - \frac{\sqrt{2}}{\rho}\left[n_1 \cosh\alpha \frac{\partial}{\partial\alpha} - \frac{\nabla_1^{\omega}}{\sinh\alpha}\right].$$

If $r^2 - t^2 > 0$, then

$$\partial_{\pm 1/2\, \pm 1/2} = -Z^{1/2}_{\pm 1/2\, \pm 1/2} \frac{\partial}{\partial\rho} + \frac{1}{\rho}[\cosh\alpha \pm \sinh\alpha \cos\vartheta] \frac{\partial}{\partial\alpha} \mp \frac{\nabla_0^{\omega}}{\rho \cosh\alpha}; \quad (31)$$

$$\partial_{-1/2\, 1/2} = \partial^*_{1/2\, -1/2} = -Z^{1/2}_{-1/2\, 1/2} \frac{\partial}{\partial\rho} + \frac{\sqrt{2}}{\rho}\left[n_1 \sinh\alpha \frac{\partial}{\partial\alpha} - \frac{\nabla_1^{\omega}}{\cosh\alpha}\right].$$

In eqs. (30) and (31), $n_1 = 2^{-1/2} \sin\vartheta e^{i\varphi}$.

In order to go the case of a Euclidean space (with coordinates $r = R\sin\alpha$ and $\tau = it = R\cos\alpha$), we need only change $\alpha$ to $i\alpha$ and $\rho$ to $-iR$ in (30).

Since the coefficient of $\partial/\partial\rho$ in $\partial_{\alpha\beta}$ is $Z^{1/2}_{\alpha\beta}$, the equalities (24) and (29) enable us to present $A_{Jl}(\rho)$ in the form

$$A_{Jl}(\rho) = \pm\left[\frac{\partial G_J(\rho)}{\partial\rho} - B_{Jl}\frac{G_J(\rho)}{\rho}\right], \quad (32)$$

where $B_{Jl}$ does not depend on $\rho$; the $\pm$ signs apply to the cases $t^2 - r^2 > 0$, and $t^2 - r^2 < 0$, respectively; $B_{Jl}$ is the same for both $t^2 - r^2 \gtrless 0$. In order to determine $B_{Jl}$, we integrate both sides of (29) over the angles $\vartheta$, $\varphi$, and use (30) or (31). If we then compare terms on right and left, we get

$$B_{Jl} = 2\varkappa(2J - 2\varkappa + 1), \quad (33)$$

$$l = J + \varkappa, \quad \varkappa = \pm 1/2.$$

Performing the operation (29) twice gives

$$L^2 Z^J_{M\mu} = 4J(J+1) Z^J_{M\mu}, \quad (34)$$

where, for the case of $t^2 - r^2 > 0$,

$$L^2 = \frac{\partial^2}{\partial\alpha^2} + 2\coth\alpha\frac{\partial}{\partial\alpha} - \frac{\hat{l}^2}{\mathrm{sh}^2\alpha}, \quad (35)$$

$$\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} - \frac{L^2}{\rho^2},$$

while for $r^2 - t^2 > 0$,

$$L^2 = \frac{\partial^2}{\partial\alpha^2} + 2\tanh\alpha\frac{\partial}{\partial\alpha} + \frac{\hat{l}^2}{\mathrm{ch}^2\alpha}, \quad (36)$$

$$\nabla^2 - \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} - \frac{L^2}{\rho^2}.$$

The Lorentz group is noncompact, and its total volume is infinite, so that it is impossible to carry out the operation of group integration using only finite-dimensional representations. It is therefore impossible to pose the question of orthogonality of $Z$-functions in pseudo-Euclidean space. In Euclidean space, the $Z$-functions are orthonormal:

$$\int_0^{\pi} d\alpha \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin^2\alpha \sin\vartheta Z^J_{M\mu} Z^{J*}_{M\mu} = \frac{2\pi^2}{2J+1}. \quad (37)$$

This enables us to expand an arbitrary function of coordinates in series in terms of the $Z^J_{M\mu}$. For example, we have for $\exp(ikx)$

$$e^{ikx} = \sum_{JM\mu} (2J+1)(-1)^J \times R_J(k\rho) Z^{J*}_{M\mu}(\beta\theta\phi) Z^J_{M\mu}(\alpha\vartheta\varphi), \quad (38)$$

or

$$e^{ikx} = \sum_{Jlm} 4\pi (-1)^J R_J(k\rho) \psi^{*}_{nlm}(\beta\theta\phi) \psi_{nlm}(\alpha\vartheta\varphi), \tag{39}$$

$$R_J(k\rho) = (2/k\rho) J_{2J+1}(k\rho),$$

where $J_n(k\rho)$ is a Bessel function, $k$ and $x$ are vectors in the four-dimensional Euclidean space whose directions are defined by the angles $\beta$, $\theta$, $\phi$ and $\alpha$, $\vartheta$, $\varphi$: $kx = |k| \rho \cos\omega$;

$$\cos\omega = \cos\beta\cos\alpha + \sin\beta\sin\alpha\cos\gamma; \tag{40}$$

$$\cos\gamma = \cos\theta\cos\vartheta + \sin\theta\sin\vartheta\cos(\varphi - \phi);$$

$$\left\{\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho^2}\frac{\partial}{\partial\rho} - \frac{L^2}{\rho^2} + k^2\right\} R_J(k\rho) Z^J_{M\mu}(\alpha\vartheta\varphi) = 0.$$

The $Z^J_{M\mu}$-functions play the same role in four-dimensional space as do the $Y_{lm}$ in three-space.

If we carry out the transformation $\alpha \to -\alpha$ or $\mathbf{r} \to -\mathbf{r}$ on $Z^{Jj}_{M\mu}$ $(\alpha, \vartheta, \varphi)$, then according to (18), for the case $t^2 - r^2 > 0$

$$Z^{Jj}_{M\mu}(-\alpha, \vartheta, \varphi) \tag{41}$$

$$= (-1)^{J+j-2\mu} Z^{Jj}_{M\mu}(\alpha, \pi - \vartheta, \pi + \varphi)$$

$$= (-1)^{\mu - M} Z^{Jj*}_{-M-\mu}(\alpha, \vartheta, \varphi),$$

and for the case $t^2 - r^2 < 0$,

$$Z^{Jj}_{M\mu}(-\alpha, \vartheta, \varphi) \tag{42}$$

$$= (-1)^{J+j} Z^{Jj}_{M\mu}(\alpha, \pi - \vartheta, \pi + \varphi)$$

$$= (-1)^{\mu + M} Z^{Jj*}_{-M-\mu}(\alpha, \vartheta, \varphi).$$

## 4. CONSTRUCTION OF RELATIVISTIC SPHERICAL SPINOR AND VECTOR FUNCTIONS.

If $\psi^{KN}_{\alpha\beta}$ transforms according to the irreducible $(2K + 1)(2N + 1)$-dimensional representation of the four-dimensional rotation group in Euclidean or pseudo-Euclidean space, then according to (12) we have the expansion

$$\psi^{KN}_{\alpha\beta} = \sum_{M,\Lambda,\mu,\lambda} C^{K\alpha}_{JML\Lambda} C^{N\beta}_{j\mu l\lambda} \psi^{Jj}_{M\mu} \psi^{Ll}_{\Lambda\lambda}, \tag{43}$$

$$\psi^{Jj}_{M\mu} \psi^{Ll}_{\Lambda\lambda} = \sum_{K,N} C^{K\alpha}_{JML\Lambda} C^{N\beta}_{j\mu l\lambda} \psi^{KN}_{\alpha\beta}.$$

For the case $L = 0$ and $l = ½$, $l = 0$ and $L = ½$, the functions $\psi^{Ll}_{\Lambda\lambda}$ become ordinary contravariant or covariant spinors.

$$\psi^{0\,1/2}_{0\sigma'}(\sigma) = \delta_{\sigma\sigma'}, \quad \psi^{1/2}_{\sigma' 0}{}^{0}(\sigma) = (-1)^{\sigma' + 1/2} \delta_{-\sigma'\sigma}. \tag{44}$$

Substituting $Z^J_{M\mu}\psi^{0\,½}_{0\sigma}$ for $\psi^{Jj}_{M\mu}\psi^{Ll}_{\Lambda\lambda}$ in (43), we form a contravariant spherical spinor function $P^{Jl}_{M\lambda}$, consisting of a pair of components ($\sigma = \pm ½$)

$$[P^{Jl}_{M\lambda}]_\sigma = \sqrt{2J+1}(-1)^J C^{l\lambda}_{J\mu - \sigma\, 1/2\, \sigma} Z^J_{M\mu}. \tag{45}$$

The factor $\sqrt{2J+1}(-1)^J$ is included in (45) to make the use of Eqs.(46) and (47) more convenient.

If we substitute $\psi^{½\,0}_{-\sigma 0}$ in (43), we get a covariant spherical spinor function $R^{Jl}_{M\lambda}$. This function is conveniently chosen so that the following equations are valid:

$$\left[\frac{\partial}{\partial t} - \vec{\sigma}\nabla\right] G_{Jl} P^{Jl}_{M\lambda} \tag{46}$$

$$= \pm i \left[\frac{\partial G_{Jl}}{\partial\rho} - 2\varkappa \frac{2J - 2\varkappa + 1}{\rho} G_{Jl}\right] R^{Jl}_{M\lambda};$$

$$\left[\frac{\partial}{\partial t} + \vec{\sigma}\nabla\right] G_{Jl} R^{Jl}_{M\lambda} \tag{47}$$

$$= \mp i \left[\frac{\partial G_{Jl}}{\partial\rho} + 2\varkappa \frac{2l + 2\varkappa + 1}{\rho} G_{Jl}\right] P^{Jl}_{M\lambda}$$

(where $l = J + \varkappa$).

The transition to a Euclidean space is made in the same way as for Eq. (30).

With the help of (29) it is not difficult to verify that

$$[R^{Jl}_{M\lambda}]_\sigma = \sqrt{2l+1}(-1)^{l+\sigma-1/2} C^{JM}_{l\lambda\, 1/2\, -\sigma} Z^l_{\Lambda\lambda}. \tag{48}$$

We shall show that the bispinor $\psi^{Jl}_{M\lambda}$ made up of $R^{Jl}_{M\lambda}$ and $P^{Jl}_{M\lambda}$, is an eigenfunction of the total four-dimensional angular momentum $\hat{J}$ of a particle with spin ½. If $\psi(\mathbf{r}, t)$ is the wave function of such a particle, then the operation of infinitesimal rotation gives

$$\psi'(\mathbf{r}+\delta\mathbf{r}, t+\delta t) \quad (49)$$

$$=\left\{1+i\vec{\delta}\left[1+\frac{1}{2}\vec{\sigma}_Д\right]-\vec{\delta}_0\left[\mathbf{g}-\frac{1}{2}\vec{\alpha}_Д\right]\right\}\psi(\mathbf{r}, t),$$

where $\vec{\delta}$ is the infinitesimal angle of the three-dimensional rotation, $\vec{\delta}_0$ is the infinitesimal velocity of the new reference system, $\vec{\sigma}_D$ and $\vec{\alpha}_D$ are the Dirac matrix vectors. We shall use the representation of the Dirac matrices in which $\gamma^5$ is diagonal. In this representation $\hat{J}^2 = L^2 + (\mathbf{l}\cdot\vec{\sigma}_D) - (\mathbf{g}\cdot\vec{\alpha}_D) + 3/2$ breaks up into two two-row matrices situated along the diagonal. Their eigenvalues are identical:

$$\hat{J}^2\psi^{Jl}_{M\lambda}=\left[4Jl+2J+2l+\frac{1}{2}\right]\psi^{Jl}_{M\lambda}. \quad (50)$$

Along with the spherical spinor functions, it is possible, by using (43), to introduce the concept of four-dimensional spherical vector functions. We shall define the contravariant and covariant spherical vectors by the relations:

$$[\mathbf{Z}^{Jl}_{M\lambda}]^{\sigma_1\sigma_2}=C^{l\lambda}_{J\mu 1\sigma}\,C^{1\sigma}_{1/2\sigma_1 1/2\sigma_2}\,Z^{J}_{M\mu}; \quad (51)$$

$$[\mathbf{Z}^{LJ}_{\Lambda\mu}]_{\sigma_1\sigma_2}=(-1)^{\sigma+1}C^{L\Lambda}_{JM1-\sigma}\,C^{1\sigma}_{1/2\sigma_1 1/2\sigma_2}\,Z^{J}_{M\mu}. \quad (52)$$

Formulas (51) and (52) are a generalization of the definition[11] of spherical vectors to the four-dimensional case.

## 5. AN EXAMPLE OF THE USE OF THE Z-FUNCTION FOR SEPARATING VARIABLES IN A RELATIVISTIC EQUATION

As an example of the use of the relativistic spherical functions, we shall consider the problem of separation of variables in the Bethe-Salpeter equation for the case of a bound state of two scalar particles. This equation can be expressed in the form

$$(\Box_1-\mu^2)(\Box_2-\mu^2)\psi(x_1, x_2) \quad (53)$$

$$=\hat{I}(x_1x_2x_3x_4)\psi(x_3, x_4),$$

where $\hat{I}$ is an integral operator. We shall denote by $D$ and $\partial$ the operators of differentiation with respect to the coordinates of the center of mass of the system $X = (x_1 + x_2)/2$ and with respect to the relative coordinates $x = x_1 - x_2$. For (53), we get

$$\{{}^1/_{16}D^4+{}^1/_2D^2\partial^2+\partial^4 \quad (54)$$

$$-{}^1/_2\mu^2D^2-2\mu^2\partial^2+\mu^4-(D\partial)^2\}\psi(X, x)$$

$$=\hat{I}(X, x, Y, y)\psi(Y, y).$$

Making use of Wick's[1] observation concerning the possibility of analytic continuation of the Bethe-Salpeter equation, we shall solve the problem in four-dimensional Euclidean space. We shall represent the wave function of a state in which the four-dimensional angular momentum of the total system has a definite value (the operator $\hat{J}^2$ is diagonal) in the form

$$\psi^J_{M\mu}=\sum_{L,\,l} g^J_{Ll}(R, \rho)\psi^J_{M\mu\,(L,\,l)}; \quad (55)$$

$$\psi^J_{M\mu\,(L,\,l)}=\sum_{\Lambda_1\lambda_1\Lambda_2\lambda_2} C^{JM}_{L\Lambda_1 l\lambda_1}\,C^{J\mu}_{L\Lambda_2 l\lambda_2}\,Z^L_{\Lambda_1\Lambda_2}(\beta\theta\phi)\,Z^l_{\lambda_1\lambda_2}(\alpha\vartheta\varphi),$$

where $l$ is the relative angular momentum of the two particles; $L$ is the angular momentum of the center of inertia relative to the origin of coordinates; $R$, $\beta$, $\theta$, $\phi$ and $\rho$, $\alpha$, $\vartheta$, $\varphi$ are the polar coordinates of $X$ and $x$; $g^J_{Ll}(R, \rho)$ is the "radial" part of the wave function. Substituting (55) in (54), we take Fourier components with respect to the coordinate $X$. We then get an equation analogous to (54), in which $D$ is replaced by the total momentum $P$ of the system, and where $G^J_{Ll}(P, \rho)\,Z^L_{\Lambda_1\Lambda_2}(\beta_p, \theta_p, \phi_p)$ appears in (55) in place of $g^J_{Ll}(R, \rho)\,Z^L_{\Lambda_1\Lambda_2}(\beta, \theta, \phi)$; $\beta_p$, $\theta_p$, $\phi_p$ are the angles of the momentum 4-vector.

The presence of the term $(P\partial)^2$ makes the equation noninvariant with respect to four-dimensional rotations in the space of the relativistic coordinates. Applying $(P\partial)^2$ to $\psi^J_{M\mu}$ gives

$$(P\partial)^2\psi^J_{M\mu}=P^2\sum_{Ll}\sum_{nk}B^{Ll}_{nk}\psi^J_{M\mu\,(L+n,\,l+k)} \quad (56)$$

$$\times\left[\frac{\partial^2}{\partial\rho^2}+\frac{3-2k(2L+k+1)}{\rho}\frac{\partial}{\partial\rho}-\frac{4l(l+1)}{\rho^2}(1-2k^2)\right]$$

$$\times G^J_{Ll}(P, \rho),$$

where $n$, $k = 0$, $\pm 1$; the coefficient $B^{Ll}_{nk}$ contains all the factors which depend only on $L$, $l$, $n$, $k$. $B^{Ll}_{nk}$ can be expressed rather simply if we make use of Racah's formula[12] for the summation of Clebsch-Gordan coefficients.

Since the $\psi^J_{M\,\mu(L,l)}$ are eigenfunctions of $\partial^2$, and since the coordinate $x$ appears in the integral operator $\hat{I}$ only in an exponential factor we can, by using the orthogonality of the $Z$-functions, obtain an infinite system of coupled equations depending on a single parameter $\rho$, for determining $G^J_{Ll}(P, \rho)$. In certain cases this system may prove to be more convenient than the initial equation (53).

Problems of separation of variables in relativistic two-body equations and possible methods for their solution will be considered in more detail in a separate paper.

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