Letters to the Editor

On Alpha-decay Theory for Nonspherical Nuclei

V. M. STRUTINSKII

(Submitted to JETP editor July 24, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 411-412 (February, 1956)

THE probability of excitation of the rotational states of a nucleus during α-decay of heavy nonspherical nuclei was calculated in the Refs. 1,2 in both of which the rotation of the nucleus was assumed to be slow compared to the motion of the α-particle (adiabatic approximation). The question whether this approximation is applicable was not really considered in either of the two works. In the case of even-even nuclei, where the moment of the total system is zero, it is possible to develop a theory which is not based on the adiabatic approximation.

The Schroedinger equation of the system α-particle + the deformed nucleus is

$$\left\{-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}} + V(\mathbf{r}) + \frac{\hbar^2}{2J}\sum_{\varkappa=\xi,\eta}(I_\varkappa - l_\varkappa)^2\right\}\Psi_I(\Omega,\mathbf{r}) = \mathcal{E}\Psi_I(\Omega,\mathbf{r}), \quad (1)$$

where the independent variables are the angle coordinates of the residual nucleus and the coordinates of the α-particle relative to the residual nucleus (vector **r**). The transformation of the Hamiltonian into new variables is most easily done by taking the corresponding classical problem into consideration. In Eq. (1), $V(\mathbf{r})$ is the interaction energy of the α-particle with the residual nucleus; $I_\varkappa$ and $l_\varkappa$ are the operators of the projection on the $\varkappa$th axis of the full moment of the system and the angular momentum of the α-particle (the axis of symmetry of the residual nucleus is chosen as the ζ axis); J is the moment of inertia of the residual nucleus about the ξ and η axis ($J_\zeta = 0$). The operators $I_\varkappa$ and $l_{\varkappa'}$ commute with each other, the operator $l_\varkappa$ acts only on the variable **r**: $l_\xi = -i\hbar(\eta\partial/\partial\zeta - \zeta\partial/\partial\eta)$ etc. Taking into account that the result of the operator $I_\varkappa$ acting on the wave function of spin 0 vanishes we shall find the equation for $\Psi_{I=0}(r)$:

$$\{-(\hbar^2/2m)\Delta_{\mathbf{r}} + V(\mathbf{r}) + \hbar^2 l^2/2J\}\Psi_0 = \mathcal{E}\Psi_0 \quad (2)$$

(the operator l_ζ^2 can be omitted as its eigenvalues vanish). Equation (2) does not contain the variables Ω; the wave function corresponding to spin 0 depends only on the relative coordinate of the α-particle. This result is fully understood, since the wave function of spin 0 should be invariant with respect to rotation of the coordinate system.

We shall develop the wave function $\Psi_0(\mathbf{r})$ in a series of Legendre polynomials:

$$\Psi_0(\mathbf{r}) = \sum_{l\,(\text{even})} \alpha_l G_l(r)\sqrt{2l+1}\,P_l(\cos\vartheta), \quad (3)$$

$$G_l(\infty) \sim r^{-1}\exp\{ik_l r\},$$

where ϑ is the angle between the direction of the vector **r** and the axis of symmetry of the nucleus. In the region outside the sphere of exit of the α-particle emerging from behind the barrier, the potential $V(\mathbf{r})$ practically does not differ from the Coulomb potential. In this region, the functions $G_l(\mathbf{r})$ fulfill the usual radial equation with energy $\mathcal{E}_l = \mathcal{E} - E_l$, where $E_l = \hbar^2 l(l+1)/2J$ is the energy of the rotational state of the daughter nucleus with spin $I' = l$, $k_l = \sqrt{2m\mathcal{E}_l}/\hbar$. We note that in the external region, at not too large l $[l \ll 2Ze^2/\hbar v(\infty)]$, one can neglect the dependence of radial function on l, so that the angular dependence of $\Psi_0(\mathbf{r})$ is the same on the surface of exit from behind the barrier and at infinity. The probability of formation of the daughter nucleus in a rotational state with spin $I' = l$ is proportional to $|\alpha_l|^2$*. Comparison with the observed intensity distribution in α-spectra of even-even nuclei with $A \geq 230$ (Ref. 3) gives the following values for the absolute values of the coefficients α_l:

$$|\alpha_0| = 1, \quad |\alpha_2|^2 \approx 0.33, \quad |\alpha_4|^2 \approx 0.003.$$

The case of $Re\,\alpha_2 > 0$ corresponds to an elongated nucleus (the absolute value of $\Psi_0(\mathbf{r})$ has a maximum for $\vartheta = 0, \pi$ and consequently the predominant direction of emission of the α-particles coincides with the axis of symmetry of the nucleus); the case when the real part of $\alpha_2 < 0$ corresponds to a flattened nucleus ($|\Psi_0(\mathbf{r})|$ is greatest for $\vartheta = \pi/2$). Under the assumption that the coefficients α_l are real, it is possible to find, from the known absolute values, the width of the angular distribution of α-particles on the exit sphere from behind the barrier. The given values $|\alpha_l|$ yield for the width $|\Psi_0(\mathbf{r})|$ the values of 100° and 88° for the elongated and the flattened nucleus respectively. The difference of the angular distribution width for the elongated and for the flattened nucleus is relatively small, and, as it will be shown below, is beyond the accuracy of the adiabatic theory. It is therefore impossible, on the basis of the adiabatic theory, to distinguish elongated and flattened nuclei from the observed intensity distribution, contrary to the claims maintained in Ref. 1.** In order to establish an actual relation between the angular distribution on the exit sphere (and consequently the intensity distribution in α-spectra) and between nuclear parameters, it is necessary to solve the Schroedinger equation (2) within the barrier region. We shall limit ourselves here only to the estimation of the nonadiabatic correction.

In the quasi-classical approximation, the Hamilton-Jacobi equation (4) corresponds to Eq. (2):

$$\frac{1}{2m}(\nabla S)^2+\frac{1}{2J}\left(\frac{\partial S}{\partial \vartheta}\right)^2=\mathcal{E}-V(\mathbf{r}), \qquad (4)$$

$$\Psi_0=\exp\left\{\frac{i}{\hbar}S(\mathbf{r})\right\}.$$

The last term on the left-hand side we shall consider to be a small perturbation. Equation (4) (without the perturbation term) is the usual Hamilton-Jacobi equation of the adiabatic approximation. Let S_0 be the action function in the adiabatic approximation. Putting $S=S_0+S'$, $|S'|\ll|S_0|$, we get the following equation for S'

$$\mathbf{v}_0\nabla S'+(1/2J)(\partial S_0/\partial\vartheta)^2=0, \text{ where } \mathbf{v}_0=m^{-1}\nabla S_0 \qquad (5)$$

with the boundary condition $S'=0$ on the surface Σ of the nucleus. The solution of Eq. (5) fulfilling the boundary condition is

$$S'=i\int_{\mathbf{r}_0}^{\mathbf{r}}\frac{1}{v_0}\frac{1}{2J}\left(\frac{\partial S_0}{\partial\vartheta}\right)^2 d\lambda. \qquad (6)$$

The path of integration is taken along the imperturbed trajectory, starting from the point $\mathbf{r}_0$ on the surface Σ. In the order of magnitude this integral equals $(L/\bar{v}_0)(\hbar^2\bar{l}^2/2J)$, ; the term added in the exponent in Eq. (4) is of the order of $\omega_0 T\bar{l}^2$, where L is the length of the path of the α-particle within the barrier, $T=L/\bar{v}_0$, $\omega_0=\hbar/2J$, , and $\bar{l}$ is the mean orbital moment of the α-particle within the barrier. In order to make the correction more exact, we have to consider the actual action function. As an example, we shall take the case of the elongated nucleus. For a sufficiently large deformation of the nucleus it is essential to assume the existence of small angles, for which the action function can be written

$$S_0\approx -i\hbar\,(a(r)-b(r)\vartheta^2), \quad b>0. \qquad (7)$$

The correction to the exponent, calculated for such an action function, amounts to $(i/\hbar)S'=4\omega_0 T\bar{b}^2\vartheta^2$. As was to be expected, the fact that the rotation is not adiabatic broadens the angular distribution of α-particles; $b_{\text{eff}}=b-4\omega_0 Tb^{-2}$. The probabilities of transition into higher rotational levels decrease correspondingly. The non-adiabatic correction may be very essential. Thus, for $L=3\times10^{-12}$ cm, $\hbar^2/2J\approx7$ kev and the adiabatic value of $b\sim5$, the correction is only slightly smaller than the main effect. The large value of the width of the angular distribution, found from the intensity distribution in α-spectra, is explained by the fact that the rotation is not adiabatic.

Remark added during proof: We have, together with G. A. Pik-Pichak, calculated the effective angle of emission ϑ^* of α-particles (of the angular width $|\varphi_0(\vec{\rho})|$) by numerical integration of the Eq. (4) for small angles of emission in elongated nuclei. If the deformation is not too small, ϑ^* does not depend to a great extent on it. Thus, for the case of the α-decay of U^{234}, ϑ^* changes from 65° for $a/b=2.4$ to 80° for $a/b=1.24$. These values are close to the "experimental" value ($\sim$ 100°). This weak dependence on the deformation of the nucleus may possibly explain the fact that the intensity distribution of the fine structure lines connected with the rotational levels is almost constant for all nuclides which are not very close to the doubly magic Pb^{208}.

* With an accuracy to terms of the order of E_l/ξ In addition, we neglect a small imaginary increment to the energy of the system.

** A straightforward answer to the question of the sign of the deformation (the sign of the quadrupole moment) can be obtained from the study of the $\alpha-\gamma$ angular correlation[4].

[1] V. G. Nosov, Dokl. Akad. Nauk SSSR **103**, 65 (1955).

[2] A. Bohr, B. Mottelson and P. Froman, Dan. Mat. Fys. Medd. **29**, 10 (1955).

[3] I. Perlman and F. Asaro, Ann. Rev. Nucl. Sci. **4**, 1954.

[4] V. M. Strutinskii. Dokl. Akad. Nauk SSSR **104**, 524 (1955).

Translated by H. Kasha,
73