ponent equal to 0.66, down to depths of 1600 meters water equivalent.

The results of our direct measurements definitely confirm George's hypothesis. However, in spite of George's assertion, one must expect the curve of the energy spectrum to start falling sharply at certain depths significantly less than 1600 meters water equivalent in the contrary case, where the mean energy of penetrating shower particles becomes more like that reported by Eidus and co-workers[12]. For this reason, further investigation of the intensity of the penetrating components of extensive cosmic ray showers at significantly greater depths is of definite interest.

In conclusion, I wish to express my thanks to Professor E. L. Andronikashvili for proposing the problem and guiding the work; to Professors C. H. Vernov and G. T. Zatsepin for valuable discussions, and to M. F. Bibilashvili, G. N. Muskhelishvili, G. E. Chikovan and G. R. Khutsishvili for assistance rendered.

[12] L. Kh. Eidus, M. P. Adamovich, I. L. Ivanovskaia, V. S. Nikolaev and M. S. Tuliankina, J. Exptl. Theoret. Phys. (U.S.S.R.) 22, 440 (1942).

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The Electromagnetic Field of a Linear Emitter Located inside an Ideally Conducting Parabolic Screen

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On the basis of a critical examination of researches previously published, the inadequate basis of the results obtained in them is pointed out. A rigorous solution of the problem is given. It is also shown that the solution takes the form corresponding to the approximation of geometrical optics in the limiting case of very high frequencies.

At first glance, the problem under consideration appears to be almost trivial, since the variables are separated in the basic equation of the problem. However, the desired choice of partial solutions, which can serve for the construction of the solution of the problem which satisfies all requirements, included the appropriate radiation principle at infinity, and the correct behavior at the focus of the cylinder, and the actual obtaining of such a solution is not so simple, as could be shown. In particular, nowhere, to our knowledge, in the series of researches[1-4] on the wave problem for a parabolic region, is any fundamental solution of the problem under consideration given in the desired form.* Such a problem is taken up in the present work. It is shown further that the solution, in the limiting case of very high frequencies, takes the form which corresponds to the ray approximation of optics.

*For more detail, see Sec. 1 of Ref. 5.

[1]Encyklopädie Math. Wissensch. 5, 3, 511.

[2]P. S. Epstein, Dissertation, Munich, 1914.

[3]W. Magnus, Jahresber. deut. Math. Verein. 50, 140 (1940).

[4]W. Magnus, Z. Physik 118, 343 (1941).

[5]G. A. Grinberg, N. N. Lebedev, I. P. Skal'skaia and Ia. S. Ufliand, Dokl. Akad. Nauk SSSR 95, 961 (1954).

1. STATEMENT OF THE PROBLEM

We consider the problem of the reflection of electromagnetic waves from a conducting screen which has the form of a parabolic cylinder. We assume a linear vibration source placed inside the cylinder along its focal line.** The current strength of the source is $I=I_0e^{i\omega t}$, where $I_0=$ const = amplitude of the current, ω=angular frequency.

We set up a cartesian coordinate system (x,y,z) so that the axis Ox lies in the plane of symmetry of the parabolic cylinder, and the axis Oz coincides with the exciting current. In addition, we introduce the parabolic coordinates (α,β) with the help of the relations

*The case of arbitrary disposition of the source is considered in Sec. 8.

$$x + iy = {}^1/_2 (\alpha + i\beta)^2, \tag{1.1}$$

$$-\infty < \alpha < +\infty, \quad 0 \leqslant \beta < \infty,$$

from which follow

$$x = {}^1/_2 (\alpha^2 - \beta^2), \quad y = \alpha\beta. \tag{1.2}$$

Then the equation of the surface of the cylinder will be $\beta = \beta_0$ (Fig. 1), where β_0 is some constant.

The electric field vector E has in this case only the single component E_z of the form

$$E_z = E(x, y)\, e^{i\omega t} = E e^{i\omega t} \tag{1.3}$$

To separate the field of the source, we write

$$E = E_0 + u, \tag{1.4}$$

where

$$E_0 = -\pi\omega I c^{-2} H_0^{(2)}(kr); \; r = \sqrt{x^2 + y^2} = {}^1/_2 (\alpha^2 + \beta^2), \tag{1.5}$$

$k = \omega/c$, c= velocity of light in a vacuum, $H_0^{(2)}(z)$ is the Hankel function of the second kind, u is the secondary electric field.

Since the electric field E must satisfy the Helm-Hotz equation $\Delta E + k^2 E = 0$, the radiation principle, and must vanish on the surface of the conducting screen, we can give the following mathematical formulation of the problem: to find the solution $E(\alpha, \beta)$ of the equation

$$(\partial^2 E / \partial\alpha^2) + (\partial^2 E / \partial\beta^2) + k^2 (\alpha^2 + \beta^2) E = 0, \tag{1.6}$$

which is regular within the parabolic cylinder, except for the line $\alpha = \beta = 0$, in the vicinity of which

$$E = -\frac{\pi\omega I}{c^2} H_0^{(2)} \left(k \frac{\alpha^2 + \beta^2}{2}\right) + u \tag{1.7}$$

(where u is a regular function), which satisfies the boundary condition

$$E|_{\beta=\beta_0} = 0 \tag{1.8}$$

and the radiation principle*

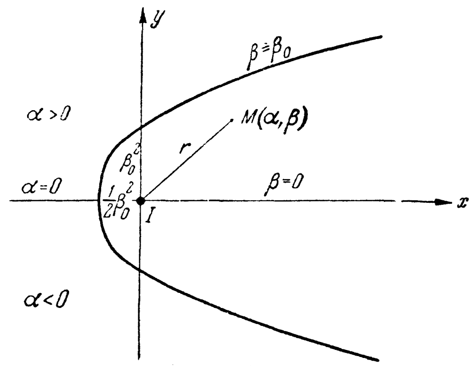

FIG. 1.

$$E|_{|\alpha| \to \infty} = O(|\alpha|^{-1/2}); \tag{1.9}$$

$$\left(\frac{1}{\alpha}\frac{\partial E}{\partial\alpha} + ikE\right)_{|\alpha| \to \infty} = o(|\alpha|^{-1/2})$$

(uniformly, relative to β).

It will be shown below (see Sec. 6) that the problem has a unique solution when stated in this way.

2. CONSTRUCTION OF A FORMAL SOLUTION OF THE PROBLEM

In order to separate variables in the wave equation

$$(\partial^2 u / \partial\alpha^2) + (\partial^2 u / \partial\beta^2) + k^2 (\alpha^2 + \beta^2) u = 0 \tag{2.1}$$

we set

$$u = A(\alpha) B(\beta). \tag{2.2}$$

We then get for the factors A and B the ordinary differential equations

$$A'' + (k^2\alpha^2 + \lambda) A = 0; \tag{2.3}$$

$$B'' + (k^2\beta^2 - \lambda) B = 0,$$

where λ is the separation parameter.

Making the substitution

$$A(\alpha) = \exp\{-ik\alpha^2/2\}\, v(\xi); \quad \xi = \sqrt{ik}\,\alpha, \tag{2.4}$$

in the first of these equations, and setting

$$\lambda = ik(2\nu + 1), \tag{2.5}$$

we get an equation for the function $v(\xi)$:

$$v'' - 2\xi v' + 2\nu v = 0, \tag{2.6}$$

the general solution of which has the form

*Equations (1.9) follow in the usual way from the condition that the same amount of energy is radiated outwards (by the wave) across an arbitrary cross section α = const per unit time, if the waves returning from infinity, are absent.

$$v(\xi) = MF\left(-\frac{\nu}{2}, \frac{1}{2}, \xi^2\right) + N\xi F\left(\frac{1-\nu}{2}, \frac{3}{2}, \xi^2\right), \tag{2.7}$$

where M and N are constants;

$$F(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!} \tag{2.8}$$

is the degenerate hypogeometric function.

Since the solution in our case must be an even function of the variable α, we must set $N=0$, so that*

$$A(\alpha) \equiv A_\nu^{(1)}(\alpha) = \exp\{-ik\alpha^2/2\} F(-\nu/2, {}^1/_2, ik\alpha^2) = (ik\alpha^2)^{-1/4} M_{\nu/2+1/4}(ik\alpha^2). \tag{2.9}$$

In this case we can limit ourselves to the interval $0<\alpha<\infty$, which is done in what follows.

If we make use of the asymptotic expression for the function $A_\nu^{(1)}(\alpha)$ as $\alpha \to \infty$,**

$$A_\nu^{(1)}(\alpha)|_{\alpha\to\infty} \approx \varphi(\nu, \alpha)\exp\{ik\alpha^2/2\}\alpha^{-(\nu+1)} + \psi(\nu, \alpha)\exp\{-ik\alpha^2/2\}\alpha^\nu, \tag{2.10}$$

where φ and ψ are functions which approach finite limits as $\alpha \to \infty$, and if we require that the desired partial solutions have the character of waves going off to infinity, and indefinitely decreasing in amplitude, then we get the condition for the parameter ν:

$$-{}^1/_2 < \mathrm{Re}\{\nu\} < 0. \tag{2.11}$$

The general integral of the second of Eqs. (2.3) (for a given value of the parameter λ) will be

$$B = B_\nu(\beta) = \exp\{-ik\beta^2/2\} \times \left[PF\left(\frac{1+\nu}{2}, \frac{1}{2}, ik\beta^2\right) + Q\sqrt{ik}\,\beta F\left(1+\frac{\nu}{2}, \frac{3}{2}, ik\beta^2\right)\right], \tag{2.12}$$

where P and Q are constants.

From the condition of boundedness of grad u at the focus it follows that $Q=0$ *, so that

$$B_\nu(\beta) \equiv B_\nu^{(1)}(\beta) = \exp\{-ik\beta^2/2\} F\left(\frac{1+\nu}{2}, \frac{1}{2}, ik\beta^2\right) \tag{2.13}$$

Thus the chosen partial solutions of the wave equation (2.1) have the form

$$u_\nu = A_\nu^{(1)}(\alpha) B_\nu^{(1)}(\beta); \quad -{}^1/_2 < R\{\nu\} < 0, \tag{2.14}$$

and the general solution of the problem can be hypothetically represented in the form of the following complex integral:

$$u(\alpha, \beta) = \int_{-\delta-i\infty}^{-\delta+i\infty} C(\nu) A_\nu^{(1)}(\alpha) B_\nu^{(1)}(\beta)\, d\nu; \quad 0<\delta<{}^1/_2. \tag{2.15}$$

The unknown function $C(\nu)$ must be found from the boundary condition (1.8) i.e., from the equality

$$\int_{-\delta-i\infty}^{-\delta+i\infty} C(\nu) A_\nu^{(1)}(\alpha) B_\nu^{(1)}(\beta_0)\, d\nu = \frac{\pi\omega I}{c^2} H_0^{(2)}\left(k\,\frac{\alpha^2+\beta_0^2}{2}\right) (-\infty<\alpha<+\infty). \tag{2.16}$$

The problem is thus reduced to an expansion of the field of the source in an integral over the functions $A_\nu^{(1)}(\alpha)$. Such an expansion can be obtained with the aid of the expansion theorem of Magnus.[3]:

$$f(\alpha) = \int_{-\delta-i\infty}^{-\delta+i\infty} F(\nu) A_\nu^{(1)}(\alpha)\, d\nu; \tag{2.17}$$

$$F(\nu) = \frac{\sqrt{k}e^{-i\pi/4}}{2\pi^2}\Gamma\left(-\frac{\nu}{2}\right) \times \Gamma\left(\frac{\nu+1}{2}\right) e^{i\pi\nu/2} \int_0^\infty f(\alpha) A_\nu^{(1)}(\alpha)\, d\alpha$$

$(0<\delta<1)$, from which it follows that

$$C(\nu) = \frac{\omega\sqrt{k}\, I e^{-i\pi/4}\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}}{2\pi c^2 B_\nu^{(1)}(\beta_0)} \times \int_0^\infty H_0^{(2)}\left(k\,\frac{\alpha^2+\beta^2}{2}\right) A_\nu^{(1)}(\alpha)\, d\alpha. \tag{2.18}$$

Making use of the value of the integral

*See, for example, Refs. 6,7.
**See Appendix A, Eq. (A-1), and also Ref. 8.

[6] W. Magnus and F. Oberheffinger, *Formeln und Satze f. d. spez. Funktionen der mathemat. Physik*, Berlin, Gottingen, Heidelberg.

[7] I. M. Ryzhik and I. S. Gradshtein, *Tables of integrals, sums, series and products*, GTTI, 1951.

[8] H. Jeffreys, *Methods of Mathematical Physics*, Cambridge, 1950.

*This result is obtained by means of a substitution of the partial solutions under consideration in Eq. (3.5) below.

$$\int_0^\infty H_0^{(2)}\left(k\frac{\alpha^2+\beta^2}{2}\right)A_\nu^{(1)}(\alpha)\,d\alpha \qquad (2.19)$$
$$= (\pi k)^{-1/2}e^{i\pi/4}B_\nu^{(3)}(\beta),$$

where*

$$B_\nu^{(3)}(\beta) = (ik\beta^2)^{-1/4}\,\Gamma\left(\frac{\nu+1}{2}\right)W_{-1/4-\nu/2,\,1/4}(ik\beta^2) \qquad (2.20)$$

($W_{\lambda,\mu}(z)$ is the Whittaker function[9]), we obtain the final solution of the problem:

$$u(\alpha,\beta) = \frac{\omega I}{2\pi\sqrt{\pi}c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right) \qquad (2.21)$$
$$\times\,\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)\,d\nu$$
$$\times\,(0<\delta<1/2).$$

It is obvious that the formal solution thus obtained needs detailed proof. Such proof is carried out in Sec. 5.

3. INVESTIGATION OF THE CONVERGENCE OF THE INTEGRAL WHICH REPRESENTS THE SECONDARY FIELD, AND PROOF OF THE REGULARITY OF THE RESULTANT SOLUTION

1. The functions under the integral in Eq. (2.21) are entire, and therefore continuous, functions of the variable ν. Since the function $B_\nu^{(1)}(\beta_0)$ does not vanish anywhere in the path of integration (see Appendix B), the existence of the integral over arbitrary finite limits is assured.

For the proof of the convergence of the integral over infinite limits, we make use of the asymptotic representations of the functions $A_\nu^{(1)}(\alpha)$, $B_\nu^{(1)}(\beta)$ and $B_\nu^{(3)}(\beta)$ for fixed α and β ($\alpha<A$, $0\leq\beta<\infty$), $|\nu|\to\infty$, $R\{\nu\}\geq-\delta>-1/2$*. Setting $\nu=-\delta+i\tau$, we have, for $\tau\to-\infty$:

$$|A_\nu^{(1)}(\alpha)| = O(1); \qquad (3.1)$$
$$|B_\nu^{(1)}(\beta)| = O(\exp\{\sqrt{2k|\tau|}\,\beta\});$$
$$|B_\nu^{(3)}(\beta)| = O(|\tau|^{-1/2}\exp\{-\sqrt{2k|\tau|}\,\beta\}).$$

Since

$$\left|\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\right| = O(|\tau|^{-1/2}), \qquad (3.2)$$

$$\left|\int_{-\delta-i\infty}^{-\delta-iT}\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\times\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)\,d\nu\right|$$
$$\leqslant M\int_T^\infty \exp\{-\sqrt{2k|\tau|}\,\beta_0\}\frac{d|\tau|}{|\tau|},$$

the uniform convergence of the integral (2.2) at the lower limit follows directly.

In precisely the same way, we find for $\tau\to+\infty$:

$$|A_\nu^{(1)}(\alpha)| = O(\exp\{\sqrt{2k\tau}\alpha\}); \qquad (3.3)$$
$$|B_\nu^{(1)}(\beta)| = O(\tau^{-1/2});$$
$$|B_\nu^{(3)}(\beta)| = O\left(\frac{1}{\sqrt{\tau}}\right);$$
$$\left|\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\right| = O(\tau^{-1/2}e^{-\pi\tau}),$$

whence

$$\left|\int_{-\delta+iT}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\times\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)\,d\nu\right| \qquad (3.4)$$
$$\leqslant M\int_T^\infty e^{-\pi\tau}e^{\sqrt{2k\tau}}\frac{d\tau}{\sqrt{\tau}}$$

so that the integral (2.21) converges uniformly at the upper limit also.

It follows from the uniform convergence of the

$$^*B_\nu^{(3)}(\beta) = \Gamma\left(\frac{1}{2}\right)\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)}B_\nu^{(1)}(\beta)+\Gamma(-1/2)B_\nu^{(2)}(\beta),$$

where

$$B_\nu^{(2)}(\beta) = e^{-ik\beta^2/2}\sqrt{ik}\,\beta F\left(1+\frac{\nu}{2},\ \frac{3}{2},\ ik\beta^2\right)$$

is the odd solution of the second of Eqs. (2.3)

*Equations (A-3) (A-4), (A-5).

[9] E. T. Whittaker and G. N. Watson, *Modern Analysis.*

integral (2.21) that u is a continuous function of the variables α and β in the region under consideration.

2. For proof of the regularity of the function u, we must satisfy ourselves of the existence and continuity of its first and second derivatives with respect to x and y, for which it is sufficient to show the unifrom convergence of the integrals which are obtained after carrying out differentiation under the integral sign.

From the estimate as $\tau \to -\infty$,

$$|A_\nu^{(1)'}(\alpha)| = O(|\tau|);$$

$$|B_\nu^{(1)'}(\beta)| = O(|\tau| \exp\{\sqrt{2k|\tau|}\beta\})$$

there follows the uniform convergence at the lower limit of the integrals obtained from Eq. (2.21) by differentiation with respect to α and β under the integral sign. In exactly the same way, there follows from $\tau \to +\infty$,

$$|A_\nu^{(1)'}(\alpha)| = O(\tau \exp\{\sqrt{2k\tau}\,\alpha\});$$

$$|B_\nu^{(1)'}(\beta)| = O(\tau)$$

the uniform convergence of these integrals at the upper limit. In a similar way, the existence and continuity of the derivatives of u of arbitrary order with respect to α and β can be demonstrated.

We now consider the derivatives of u with respect to x and y, which are given by the expressions

$$\frac{\partial u}{\partial x} = \frac{\alpha}{\alpha^2+\beta^2}\frac{\partial u}{\partial \alpha} - \frac{\beta}{\alpha^2+\beta^2}\frac{\partial u}{\partial \beta}; \tag{3.5}$$

$$\frac{\partial u}{\partial y} = \frac{\alpha}{\alpha^2+\beta^2}\frac{\partial u}{\partial \beta} + \frac{\beta}{\alpha^2+\beta^2}\frac{\partial u}{\partial \alpha}.$$

It follows from these equations that $\partial u/\partial x$ and $\partial u/\partial y$ are continuous everywhere in the region under consideration, with the exception, perhaps, of the point $\alpha=\beta=0$, in the vicinity of which an additional investigation is necessary. We set

$$\alpha = r\cos\theta;\ \ \beta = r\sin\theta\ (0 \leqslant \theta \leqslant \pi) \tag{3.6}$$

and consider the expression

$$\frac{\partial u}{\partial x} = \frac{\omega I}{2\pi\sqrt{\pi}c^2}\int_{-\delta-i\infty}^{-\delta+i\infty} \Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}\chi_\nu(r)\,d\nu, \tag{3.7}$$

where

$$\chi_\nu(r) = \frac{1}{r}\left[\cos\theta\, B_\nu^{(1)}(r\sin\theta)\, A_\nu^{(1)'}(r\cos\theta) - \sin\theta\, A_\nu^{(1)}(r\cos\theta)\, B_\nu^{(1)'}(r\sin\theta)\right]. \tag{3.8}$$

Since the quantity $\chi_\nu(r)$ approaches a finite limit as $r \to 0$, equal to $-ik(2\nu+1)$, and uniform convergence of the integral (3.7) in the region $r \leqq r_0$ follows from the estimate

$$|\chi_\nu(r)|_{|\tau|\to\infty} = O(|\tau|\exp\{\sqrt{2k|\tau|}\,r_0\}),$$

we can allow a transition to the limit under the integral sign. Then

$$\lim_{r\to 0}\frac{\partial u}{\partial x} = -\frac{ik\omega I}{2\pi\sqrt{\pi}c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}(2\nu+1)\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}\,d\nu.$$

This integral does not depend on θ, which also proves the continuity of the derivative $\partial u/\partial x$ in the vicinity of the point $\alpha=\beta=0$.

As regards the limit of the derivative $\partial u/\partial y$ for

$r \to 0$, the analogous expressions show that $\lim_{r\to 0} \partial u/\partial y = 0$, i.e., the derivative is also continuous in the vicinity of the point. In similar fashion we can establish the continuity of the second derivatives of u with respect to x and y.

Hence it has been shown that $u(\alpha, \beta)$ is a regular function in the entire region $\alpha < A$, $0 \leq \beta \leq \beta_0$.

4. TRANSFORMATION OF THE SOLUTION TO DIFFERENT FORMS

For subsequent examination of the formal solution that has been obtained, it is expedient first to transform it to other forms.

1. In order to obtain the first of these representations, we complete the contour of integration in the formula (2.21) on the right by another circle of radius $R \to \infty$, drawn from the point $\nu = -1/2$. and apply the theorem of residues. The asymptotic expressions for the functions of a parabolic cylinder* show that the integral over the path, under fulfillment of the condition

$$\alpha < 2\beta_0 - \beta \qquad (4.1)$$

in every case tends to zero.

If the inequality (4.1) holds then u, after calculation of the residues at the poles $\nu = 2n$, has the form of the following series:

$$u = \frac{2\omega i I}{c^2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} \frac{B_{2n}^{(3)}(\beta_0)}{B_{2n}^{(1)}(\beta_0)} B_{2n}^{(1)}(\beta) A_{2n}^{(1)}(\alpha). \qquad (4.2)$$

The corresponding transformation of the contour integral (2.16)-(2.18) for the primary field reduces to the analogous expression

$$H_0^{(2)}\left(k\frac{\alpha^2+\beta^2}{2}\right) = \frac{2i}{\pi^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} B_{2n}^{(3)}(\beta) A_{2n}^{(1)}(\alpha), \qquad (4.3)$$

which is valid under the condition that

$$\alpha < \beta. \qquad (4.4)$$

We note that the functions $A_{2n}^{(1)}(\alpha)$ are simply related to the Hermite polynomials:

$$A_{2n}^{(1)}(\alpha) = (-1)^n \frac{n!}{(2n)!} \exp\{-ik\alpha^2/2\} H_{2n}(\sqrt{ik}\,\alpha). \qquad (4.5)$$

Thus, our solutions for the case $\alpha < \beta$ can always be represented in the form of a series in the Hermite polynomials $H_{2n}((\sqrt{ik}\,\alpha).)/.$: In regard to the region $\alpha > \beta$, we can demonstrate, by direct investigation of the convergence of the series (4.3) that it converges for $\alpha > \beta$. Similarly, the series (4.2) is convergent* for $\alpha > 2\beta_0 - \beta$. Summing up, we can say that the decomposition of the total field into a series of Hermite polynomials is valid only in those parts of the region which are determined by the inequality $\alpha < \beta$. Thus, there does not exist a single analytical representation of the desired solution in the form of a series of Hermite polynomials.**

2. A solution of the problem has been given by Magnus[4] in the form

$$u = \frac{\omega I}{2\pi c^2} \int_{-\delta - i\infty}^{-\delta + i\infty} \frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)} B_\nu^{(1)}(\beta) A_\nu^{(3)}(|\alpha|)\, d\nu \quad (0 < \delta < 1/2), \qquad (4.6)$$

where

$$A_\nu^{(3)}(\alpha) = B_{-\nu-1}^{(3)}(\beta). \qquad (4.7)$$

Inasmuch as Magnus did not establish the regularity of his solution for an arbitrary point within the region of interest, it is not possible, without further investigation, to draw any conclusions as to the correctness of his solution. A detailed investigation is all the more necessary in that the partial solutions $A_\nu^{(3)}(|\alpha|)\beta_\nu^{(1)}(\beta)$ given by Magnus do not possess the property of regularity in the vicinity of the point $\alpha = \beta = 0$.

Coincidence of the solution of Magnus (4.6) with our solution (2.21) in there common region of regularity can be established by means of the addition of an integration contour in (4.6) on the right of another circle of radius $R \to \infty$ (which is permissible for $\alpha < 2\beta_0 - \beta$) and computation of the residues at the poles $\nu = 2n$. In this case, since the

*See Eqs. (A-3) - (A-5).

*The convergence of these series has been investigated with the help of the asymptotic expressions for the Hermite polynomials for $n \to \infty$.

*This circumstance has evidently not been noted properly in the literature.

previous formula (4.2) is obtained, then, by virtue of the known properties of the solution of equations of the elliptic type, the identity of the solutions (4.6) and (4.2) takes place wherever the two solutions are regular.

We also note that the field of the source can also be decomposed into integrals of the functions $A_\nu^{(3)}(|\alpha|)$:

$$H_0^{(2)}\left(k\frac{\alpha^2+\beta^2}{2}\right)=\frac{1}{2\pi^2}\int_{-\delta-i\infty}^{-\delta+i\infty} B_\nu^{(3)}(\beta)A_\nu^{(3)}(|\alpha|)\,d\nu \quad (0<\delta<1), \tag{4.8}$$

this decomposition is valid in all regions, with the exception of the line $\beta=0$.*

3. On the basis of the results obtained in Sec. 2, the complete solution E can be represented by the following integral**

$$E=\frac{\omega I}{2\pi c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}[B_\nu^{(1)}(\beta_0)B_\nu^{(2)}(\beta) \tag{4.9}$$

$$-B_\nu^{(2)}(\beta_0)B_\nu^{(1)}(\beta)]\frac{A_\nu^{(3)}(|\alpha|)}{B_\nu^{(1)}(\beta_0)}d\nu \quad (\beta>0),$$

which allows us to obtain still another form of the solution, which is extremely useful for what follows.

We complete the contour of integration with another circle $R\to\infty$ at the left, and make use of the theorem of residues. Since the poles of the integrand are the roots of equation

$$B_{\nu_n}^{(1)}(\beta_0)=0 \quad (n=1,\ 2,\ 3,\ \ldots), \tag{4.10}$$

located, as shown in Appendix B, to the right of $Re\{\nu\}=-1/2$, where $\mathrm{Im}\{\nu_1\}<\mathrm{Im}\{\nu_2\}<\ldots$, and the integral along the path approaches zero, we obtain

$$E=-\frac{2\sqrt{\pi}\,ikI}{c}\sum_{n=1}^{\infty}\frac{B_{\nu_n}^{(2)}(\beta_0)B_{\nu_n}^{(1)}(\beta)}{[dB_\nu^{(1)}(\beta_0)/d\nu]_{\nu=\nu_n}}A_{\nu_n}^{(3)}(|\alpha|). \tag{4.11}$$

Calculation of the derivative in the square brackets leads to the following decomposition of E in a series of functions $B_{\nu n}^{(1)}(\beta)$:

$$E(\alpha,\ \beta)=-\frac{I\sqrt{\pi}\,ik}{c} \tag{4.12}$$

$$\times\sum_{n=1}^{\infty}B_{\nu_n}^{(1)}(\beta)A_{\nu_n}^{(3)}(|\alpha|)\Big/\int_0^{\beta_0}B_{\nu_n}^{(1)2}(\beta)\,d\beta.$$

Such a form of the solution will be used in the following Section for proof of the fact that the solution of the problem satisfies the boundary condition and the radiation principle.

We also note that the decomposition (4.12) can be obtained directly by integration of the wave equation

$$(\partial^2E/\partial\alpha^2)+(\partial^2E/\partial\beta^2) \tag{4.13}$$

$$+k^2(\alpha^2+\beta^2)E=4\pi i\omega c^{-2}(\alpha^2+\beta^2)j,$$

if we consider the current density to be uniformly distributed over a small region about the focus:

$$j(\alpha,\ \beta)$$

$$=\begin{cases}I/2(\alpha^2+\beta^2)\alpha_1\beta_1 & \text{for } |\alpha|<\alpha_1,\quad 0\leqslant\beta<\beta_1,\\ 0 \text{ for other values of } \alpha \text{ and } \beta,\end{cases}$$

while, in what follows, α,β, approach zero.

5. PROOF OF THE EQUATION OBTAINED FORMALLY

It follows from the results of Sec. 3 that the function u constructed above, which is defined by Eq. (2.21), is a regular solution of the Helmhotz equation at every point of the region under examination. As far as the complete solution E is concerned, this, being the sum of the field of the linear radiator which proceeds across the focus, and the function u, it evidently satisfies the Helmhotz equation at all points of the region except the focus, where it has a singularity of the required type.

It is then left to prove that the solution $E(\alpha,\beta)$ satisfies the boundary condition and the radiation

*See Ref. 4.

*Here we make use of the relation $B_\nu^{(3)}(\beta_0)B_\nu^{(1)}(\beta)-B_\nu^{(1)}(\beta_0)B_\nu^{(3)}(\beta)=2\sqrt{\pi}[B_\nu^{(1)}(\beta_0)B_\nu^{(2)}(\beta)-B_\nu^{(2)}(\beta_0)B_\nu^{(1)}(\beta)]$

principle. Such a proof can be carried out with the help of the form of Equation (4.12).

1. Since the eigenfunctions $B_{\nu n}^{(1)}(\beta)$ satisfy the boundary condition $B_{\nu n}^{(1)}(\beta_0)=0$, to prove that the boundary condition $E(\alpha,\beta_0)=0$ is satisfied, it suffices to establish the uniform convergence of the series (4.12) in the interval $0 \leq \beta \leq \beta_0$.

Making use of the asymptotic expressions for the functions $B_\nu^{(1)}(\beta)$ and $A_\nu^{(3)}$ for $\nu_n = -1/2 + i\tau_n$ ($\tau_n \to \infty$):

$$B_{\nu_n}^{(1)}(\beta) \approx \cos\sqrt{2ki\tau_n}\,\beta; \tag{5.1}$$

$$A_{\nu_n}^{(3)}(\alpha) \approx 2\sqrt{2\pi i/\tau_n}\exp\{-\sqrt{2k\tau_n}\,\alpha\},$$

and also Eq. (B-2)

$$\tau_n\Big|_{n\to\infty} \approx \frac{1}{2k\beta_0^2}\left(\frac{2n-1}{2}\pi\right)^2, \tag{5.2}$$

we find that the terms of the series (4.12) will be of the order

$$n^{-1}\exp\left\{-\frac{2n-1}{2\beta_0}\pi\alpha\right\},$$

from which follows the uniform convergence of the series in the region of interest. At the same time, satisfaction of the boundary condition is proved.

2. In order to demonstrate that the solution we have obtained satisfies the radiation principle (1.9), it suffices to show that the following two equations are satisfied:

$$|u|_{\alpha\to\infty} = O(\alpha^{-1/2}); \tag{5.3}$$

$$\left|\frac{1}{\alpha}\frac{\partial u}{\partial \alpha} + iku\right|_{\alpha\to\infty} = o(\alpha^{-1/2}).$$

From the estimate for $n \to \infty$,

$$\left|\frac{B_{\nu_n}^{(1)}(\beta)}{\int_0^{\beta_0} B_{\nu_n}^{(1)2}(\beta)\,d\beta}\right| = O(1),\ |A_{\nu_n}^{(3)}(\alpha)| = \alpha^{-1/2}O(n^{-3/2}), \tag{5.4}$$

the correctness of the first part of the radiation principle follows at once.

*The second estimate of (5.4), which is investigated in Appendix C, is uniform relative to α.

Differentiating the series (4.12) and making use of the estimate

$$\left|\frac{1}{\alpha}A_{\nu_n}^{(3)\prime}(\alpha) + ikA_{\nu_n}^{(3)}(\alpha)\right|_{n\to\infty} \tag{5.5}$$

$$= \alpha^{-5/2}O(n^{-3/2}),$$

investigated in Appendix C, which is valid for arbitrary α, we have the equality

$$\left|\frac{1}{\alpha}\frac{\partial u}{\partial \alpha} + iku\right|_{\alpha\to\infty} = O(\alpha^{-5/2}),$$

whereby the second part of the radiation principle is established.

Thus the solution of the problem obtained formally in Sec. 2 satisfies all the conditions set forth in Sec. 1 in its mathematical formulation.

6. UNIQUENESS THEOREM

The solution of the Helmholtz equation

$$\Delta v + k^2 v = 0, \tag{6.1}$$

is regular in the region (D), bounded by the branch of the parabola $\beta=\beta_0$ and the abscissa, satisfies the boundary conditions

$$v|_{\beta=\beta_0} = 0,\ \partial v/\partial y|_{y=0} = 0 \tag{6.2}$$

and the radiation principle**

$$v\Big|_{\alpha\to\infty} = O(\alpha^{-1/2}); \tag{6.3}$$

$$\frac{1}{\alpha}\frac{\partial v}{\partial \alpha} + ikv\Big|_{\alpha\to\infty} = o(\alpha^{-1/2}),$$

is identically equal to zero.

For proof, we consider the region (D'), bounded by the x axis, the branch of the parabola $\beta=\beta_0$ and the branch of the parabola $\alpha=A$ orthogonal to it. Multiplying Eq. (6.1) by the conjugate function $\bar{v}$, and the conjugate function by v, calculating the one from the other and integrating over the area (D'), we obtain [making use of Green's theorem and the condition (6.2)] :

**And similarly for the variable β.

$$\int_0^{\beta_0}\left(\bar{v}\frac{\partial v}{\partial\alpha} - v\frac{\partial\bar{v}}{\partial\alpha}\right)_{\alpha=A} d\beta = 0. \tag{6.4}$$

Making use of the relation

$$\frac{1}{\alpha}\left(\bar{v}\frac{\partial v}{\partial\alpha} - v\frac{\partial\bar{v}}{\partial\alpha}\right) \tag{6.5}$$

$$= -2ik|v|^2 + \bar{v}o(\alpha^{-1/2}) - vo(\alpha^{-1/2})$$

which follows from the radiation principle, and taking into consideration the fact that as $\alpha\to\infty$ the products $\alpha^{1/2}v$ and $\alpha^{1/2}\bar{v}$ are bounded quantities, we obtain the following equality from Eq. (6.4):

$$\lim_{\alpha\to\infty}\int_0^{\beta_0}\alpha|v|^2 d\beta = 0. \tag{6.6}$$

In order to show that the identity $v=0$ follows from Eq. (6.6), we consider the following partial solution of the Helmholtz equation:

$$u_n = A^{(3)}_{\nu_n}(\alpha)\, B^{(1)}_{\nu_n}(\beta), \tag{6.7}$$

where ν_n are the roots of the equation $B^{(1)}_{\nu n}(\beta_0) = 0$. These solutions are regular in the region (D''), which is bounded by the parabola $\beta=\beta_0$, $\alpha=\epsilon>0$, $\alpha=A$ and the x axis, and satisfy the boundary conditions

$$u_n|_{\beta=\beta_0} = 0;\ \partial u_n/\partial\beta|_{\beta=0} = 0 \tag{6.8}$$

and the radiation principle (6.3).

Applying Green's formula to the region (D'') for the pair of functions (v, u_n), we get

$$\int_0^{\beta_0}\left(u_n\frac{\partial v}{\partial\alpha} - v\frac{\partial u_n}{\partial\alpha}\right)_{\alpha=\varepsilon} d\beta \tag{6.9}$$

$$= \int_0^{\beta_0}\left(u_n\frac{\partial v}{\partial\alpha} - v\frac{\partial u_n}{\partial\alpha}\right)_{\alpha=A} d\beta.$$

From this it follows that the value of the integral

$$\int_0^{\beta_0}\left(u_n\frac{\partial v}{\partial\alpha} - v\frac{\partial u_n}{\partial\alpha}\right) d\beta$$

does not depend on α, and, since as $\alpha\to\infty$ this integral approaches zero (on the basis of the radiation principle), then

$$\int_0^{\beta_0}\left(u_n\frac{\partial v}{\partial\alpha} - v\frac{\partial u_n}{\partial\alpha}\right) d\beta = 0 \text{ for arbitrary } \alpha. \tag{6.10}$$

Substituting here the value of u_n, and assuming

$$v_n = \int_0^{\beta_0} vB^{(1)}_{\nu_n}\,d\beta \Big/ \int_0^{\beta_0} B^{(1)^2}_{\nu_n}\,d\beta \quad (n = 1, 2, 3, \ldots), \tag{6.11}$$

we find

$$A^{(3)}_{\nu_n}(\alpha)\frac{dv_n}{d\alpha} = v_n\frac{dA^{(3)}_{\nu_n}(\alpha)}{d\alpha},$$

whence

$$v_n = c_n A^{(3)}_{\nu_n}(\alpha), \tag{6.12}$$

where c_n is a constant.

Since the system of functions $B_{\nu n}(\beta)$ is closed*, i.e.,

$$\sum_{n=1}^{\infty}|v_n|^2\int_0^{\beta_0} B^{(1)^2}_{\nu_n} d\beta = \int_0^{\beta_0}|v|^2 d\beta, \tag{6.13}$$

then it follows from Eq. (6.6) that $\lim_{\alpha\to\infty}\alpha|v_n|^2=0$ or, on the basis of Eq. (6.12),

$$|c_n|^2\lim_{\alpha\to\infty}\alpha|A^{(3)}_{\nu_n}|^2 = 0. \tag{6.14}$$

But the asymptotic expression for the function $A^{(3)}_{\nu n}(\alpha)$ as $\alpha\to\infty$** shows that the product $\alpha|A^{(3)}_{\nu n}(\alpha)|^2$ does not approach zero as $\alpha\to\infty$, so that $c_n = 0$, i.e.,

$$v_n = 0;\ n = 1, 2, 3, \ldots \tag{6.15}$$

By virtue of the closure condition (6.13), we have $v=0$ for arbitrary $\alpha>0$ and $0\leq\beta\leq\beta_0$, which also was to be proved.

7. INVESTIGATION OF THE SOLUTION FOR LARGE VALUES OF THE WAVE NUMBER *k*. TRANSITION TO GEOMETRIC OPTICS

We shall investigate the behavior of the solu-

*This follows from the general theory of eigenfunctions associated with differential equations of second order.

**See Eq. (A-2).

tion obtained in Sec. 2 as $k\to\infty$, which should correspond to the case of geometric optics. To investigate the behavior as $k\to\infty$ of the secondary field, we make use of the following asymptotic formulas:

$$B_\nu^{(1)}(\beta)\,|_{k\to\infty}\approx\frac{\sqrt{\pi}}{\Gamma((\nu+1)/2)}(ik\beta^2)^{\nu/2}e^{ik\beta^2/2};\tag{7.1}$$

$$B_\nu^{(3)}(\beta)\,|_{k\to\infty}\approx\Gamma\left(\frac{\nu+1}{2}\right)(ik\beta^2)^{-(1+\nu)/2}e^{-ik\beta^2/2},\quad R\{\nu\}>-{}^1/_2,$$

which were obtained from Eqs. (A-1) and (A-2) by replacing ν by $-\nu-1$, taking the relation $A_\nu(\alpha)=B_{-\nu-1}(\alpha)$ into account.

Substitution of (7.1) into (2.21) leads to the expression:

$$u(\alpha,\beta)\,|_{k\to\infty}\approx\frac{I\sqrt{k}\,e^{-i\pi/4}}{2\pi\sqrt{\pi}\,c\beta_0}\exp\left\{\frac{ik}{2}(\beta^2-2\beta_0^2)\right\}I_1;\tag{7.2}$$

$$I_1=\int\limits_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)k^{-\nu/2}\left(\frac{\beta_0^2}{\beta}\right)^{-\nu}e^{i\pi\nu/4}A_\nu^{(1)}(\alpha)\,d\nu.$$

In calculating the integral I_1 we make use of the integral representation (A-6) which, when substituted in Eq. (7.2), yields

$$I_1=\frac{\sqrt{2k}}{c\beta_0}\exp\left\{-\frac{i\pi}{4}-\frac{ik}{2}(\beta^2-2\beta_0^2)\right\}\int\limits_{-\infty}^{\infty}\exp\left\{\frac{\theta}{2}-\frac{ik\alpha^2}{2}\,\mathrm{th}\,\theta\right\}\frac{I_2d\theta}{\sqrt{\mathrm{ch}\theta}};\tag{7.3}$$

$$I_2=\frac{1}{2\pi 1}\int\limits_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(\frac{\nu+1}{2}\right)e^{\nu\theta}k^{-\nu/2}\left(\frac{\beta_0^2}{\beta}\right)^{-\nu}e^{i\pi\nu/4}d\nu.$$

We make the substitution $\nu=2\mu-1$ in the complex integral. Denoting $2\delta'=1-\delta$, we have

$$I_2=\frac{2\beta_0^2}{\sqrt{k\beta}}\exp\left(-\frac{i\pi}{4}-\theta\right)\frac{1}{2\pi i}\int\limits_{\delta'-i\infty}^{\delta'+i\infty}\Gamma(\mu)\left(-ike^{-2\theta}\frac{\beta_0^4}{\beta^2}\right)^{-\mu}d\mu,$$

whence*

$$I_2=2\sqrt{k}\left(\frac{\beta_0^2}{\beta}\right)\exp\left\{-\frac{i\pi}{4}-\theta+ik\frac{\beta_0^4}{\beta^2}e^{-2\theta}\right\}.\tag{7.4}$$

Substituting (7.4) in (7.3), and carrying out the substitution $e^{-\theta}=x$ in the integral I_1, we get

$$u\,|_{k\to\infty}\approx\frac{4kI\beta_0}{c\beta}\exp\left\{\frac{ik}{2}(\beta^2-2\beta_0^2)\right\}\int\limits_0^\infty\tag{7.5}$$

$$\times\exp\left\{-\frac{ik\alpha^2}{2}\frac{1-x^2}{1+x^2}+ik\frac{\beta_0^4}{\beta^2}x^2\right\}\frac{dx}{\sqrt{1+x^2}}.$$

As $k\to\infty$ the principal contribution to the latter integral comes from the neighborhood of the point $x=0$. Therefore

$$u|_{k\to\infty}=\frac{1}{\sqrt{c}}2I(\pi k)^{1/2}[\beta_0^2+(\alpha\beta/\beta_0)^2]^{-1/2}\tag{7.6}$$

$$\times\exp\left\{\frac{i\pi}{4}-ik\beta_0^2-\frac{ik}{2}(\alpha^2-\beta^2)\right\}.$$

Transforming to cartesian coordinates, and taking into account the relation $s=OA+AM=1/2(\alpha^2+\beta^2)+\beta_0^2$(Fig. 2), we can transform Eq.(7.6) to the form

$$u\,|_{k\to\infty}\approx\frac{2I}{c\beta_0}\left(\frac{\pi k}{1+(y/\beta_0)^2}\right)^{1/2}e^{i\pi/4-iks}.\tag{7.7}$$

It is immediately evident that the index s, which appears in the phase factor, represents the path OAM which that ray would follow which sets out

*Here we make use of the relation

$$e^{-x}=\frac{1}{2\pi i}\int\limits_{\delta'-i\infty}^{\delta'+i\infty}\Gamma(\mu)x^{-\mu}d\mu.$$

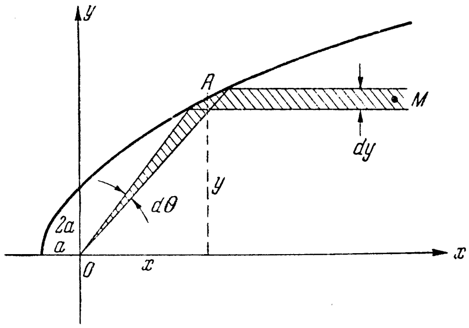

FIG. 2.

from the focus O and falls on the point M as a result of reflection (according to the laws of geometric optics) from the surface of the parabolic cylinder.

So far as the amplitude factor in Eq. (7.7) is concerned, we can demonstrate that it also corresponds to the approximation of geometric optics. In fact, computing the mean over a period, of the density p of energy flow incident at a certain point M** :

$$p = (c / 16\pi)(u^* H_y - u H_y^*), \qquad (7.8)$$

and using the asymptotic representations for H_y [coincident with Eq. (7.7)], we get

$$p|_{k\to\infty} \approx (kI^2/2c\beta_0^2)/[1 + (y/\beta_0^2)^2], \qquad (7.9)$$

On the other hand, we can get this same expression by starting out with the laws of geometric optics, if we calculate the electromagnetic energy $\frac{W}{2\pi} d\theta = \frac{W}{2\pi |dy/d\theta|} |dy| = p|dy|$, which is transmitted per unit time across an area of height dy in the vicinity of the point M (Fig. 2); here, $W = \pi k I^2/2c$ denotes the time average of the amount of energy radiated by the source per unit length. From the geometric relations $\tan \theta = y/x$, $y^2 = 2\beta_0^2(x + \beta_0^2/2)$, we find that

$$y = \beta_0^2 \operatorname{ctg} \frac{\theta}{2}; \quad \frac{dy}{d\theta} = -\frac{\beta_0^2}{2}\left[1 + \left(\frac{y}{\beta_0}\right)^2\right],$$

whence the above expression (7.9) is obtained for the quantity p.

Thus the solution of the problem in the limiting case k transforms us to the approximation of geometric optics.

**Here we neglect the component H_x of the secondary magnetic field, since it is of the order of $1/k$ in comparison with H_y.

8. GENERAL SOLUTION IN THE CASE OF AN ARBITRARY LOCATION OF THE SOURCE ON THE AXIS OF THE CYLINDER

When the source of vibrations is not placed at the focus, the method of solution of the problem set forth in Sec. 2 leads to complicated calculations, which are concerned with the decomposition of the source field in the integral into parabolic cylindrical functions. In such problems, it is expedient to proceed otherwise, namely, by investigating the total field at once.

Let us first consider the case in which the source is located at an arbitrary point $\alpha = \alpha^*$, $\beta = 0$ on the positive part of the abscissa. Replacing the linear current I by a current which is equal to it but uniformly distributed over a small area, we can write down the following inhomogeneous equation:

$$\frac{\partial^2 E}{\partial \alpha^2} + \frac{\partial^2 E}{\partial \beta^2} + k^2(\alpha^2 + \beta^2) E = \frac{4\pi i \omega}{c^2}(\alpha^2 + \beta^2) j(\alpha, \beta)$$

$$(0 < \alpha < \infty, \quad 0 < \beta < \beta_0);$$

$$j(\alpha, \beta) = \begin{cases} I/[4(\alpha^2 + \beta^2)\varepsilon\gamma] & \text{for } \alpha^* - \varepsilon < \alpha < \alpha^* + \varepsilon, \ 0 < \beta < \gamma, \\ 0 & \text{for other } \alpha \text{ and } \beta, \end{cases} \qquad (8.1)$$

and the boundary conditions

$$E|_{\beta=\beta_0} = 0, \quad \partial E/\partial\alpha|_{\alpha=0} = 0, \quad \partial E/\partial\beta|_{\beta=0} = 0. \qquad (8.2)$$

If we seek the function $E(\alpha, \beta)$ in the form of an expansion

$$E = \frac{\sqrt{k}\, e^{-i\pi/4}}{2\pi^2} \int_{-\delta-i\infty}^{-\delta+i\infty} E_\nu(\beta)\, \Gamma\left(-\frac{\nu}{2}\right) \qquad (8.3)$$

$$\times\ \Gamma\left(\frac{\nu+1}{2}\right) e^{i\pi\nu/2} A_\nu^{(1)}(\alpha)\, d\nu, \quad 0 < \delta < 1/2,$$

where

$$E_\nu(\beta) = \int_0^\infty E A_\nu^{(1)}(\alpha)\, d\alpha, \qquad (8.4)$$

then, multiplying (8.1) by $A_\nu^{(1)}(\alpha)$, and integrating* over α from zero to infinity, we get the usual differential equation for $E_\nu(\beta)$. Solving it under the conditions $E_\nu(\beta_0) = 0$, $E_\nu'(0) = 0$, we are led to the expression

$$E_\nu(\beta) = \frac{I\sqrt{\pi i k}}{c} B_\nu^{(1)}(\beta) B_\nu^{(3)}(\beta_0) \qquad (8.5)$$

$$- B_\nu^{(3)}(\beta) B_\nu^{(1)}(\beta_0) \frac{A_\nu^{(1)}(\alpha^*)}{B_\nu^{(1)}(\beta_0)}.$$

*Here we also make use of the radiation principle.

as ϵ, γ, $\to 0$. Substituting (8.5) in (8.3), we obtain the final solution of the problem:

$$E(\alpha, \beta) = \frac{Ik}{2\pi\sqrt{\pi}\, c} \int_{-\delta-i\infty}^{-\delta+i\infty} \Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right) e^{i\pi\nu/2}\,[B_\nu^{(1)}(\beta)\, B_\nu^{(3)}(\beta_0) - B_\nu^{(3)}(\beta)\, B_\nu^{(1)}(\beta_0)]\,\frac{A_\nu^{(1)}(\alpha^*)}{B_\nu^{(1)}(\beta_0)}\, A_\nu^{(1)}(\alpha)\, d\nu. \quad (8.6)$$

In a similar fashion, we can consider the case in which the source is located to the left of the focus at an arbitrary point on the negative abscissa.

We have limited ourselves here to the consideration of the case in which the source lies on the axis of the cylinder. Similarly, it would be possible to investigate the general case of an arbitrary location of the source within the cylinder. In such a case, we would have to take into consideration (along with the four functions $A_\nu^{(1)}(\alpha)$ odd functions, for which one would have to introduce the corresponding expansion formula.

APPENDIX A
SOME PROPERTIES OF PARABOLIC CYLINDRICAL FUNCTIONS
ASYMPTOTIC BEHAVIOR FOR LARGE VALUES OF THE ARGUMENT:

$$A_\nu^{(1)}(\alpha)\big|_{|\alpha|\to\infty} = e^{ik\alpha^2/2}\frac{\Gamma(1/2)}{\Gamma(-\nu/2)}(ik\alpha^2)^{-(\nu+1)/2}\left[1 + O\left(\frac{1}{k\alpha^2}\right)\right] + \exp\left\{-\frac{ik\alpha^2}{2} - \frac{i\pi\nu}{2}\right\}\frac{\Gamma(1/2)}{\Gamma[(\nu+1)/2]}(ik\alpha^2)^{\nu/2}\left[1 + O\left(\frac{1}{k\alpha^2}\right)\right]; \quad \text{(A-1)}$$

$$A_\nu^{(3)}(\alpha)\big|_{|\alpha|\to\infty} = (ik\alpha)^{\nu/2}\,\Gamma\left(-\frac{\nu}{2}\right) e^{-ik\alpha^2/2}\left[1 + O\left(\frac{1}{k\alpha^2}\right)\right]. \quad \text{(A-2)}$$

Asymptotic representation for $|\nu| \to \infty$:

$$A_\nu^{(1)}(\alpha) = \cos\sqrt{ik(2\nu+1)}\,\alpha\left[1 + O\left(\frac{1}{\sqrt{2\nu+1}}\right)\right]; \quad \text{(A-3)}$$

$$B_\nu^{(1)}(\beta) = \operatorname{ch}\sqrt{ik(2\nu+1)}\,\beta\left[1 + O\left(\frac{1}{\sqrt{2\nu+1}}\right)\right]; \quad \text{(A-4)}$$

$$B_\nu^{(3)}(\beta) = \mp\frac{2\sqrt{\pi}}{\sqrt{2\nu+1}}\exp\{\pm\sqrt{ik(2\nu+1)}\,\beta\}. \quad \text{(A-5)**}$$

Integral representations

$$A_\nu^{(1)}(\alpha) = \frac{\sqrt{2\pi}}{\Gamma(-\nu/2)\Gamma((\nu+1)/2)}\int_{-\infty}^{\infty}\exp\left\{\left(\nu+\frac{1}{2}\right)\theta - \frac{ik\alpha^2}{2}\operatorname{th}\theta\right\}\frac{d\theta}{\sqrt{\operatorname{ch}\theta}} \quad (-1 < \operatorname{Re}\nu < 0); \quad \text{(A-6),***}$$

$$A_\nu^{(3)}(\alpha) = \frac{2^{\nu+1}\sqrt{\pi}}{\Gamma((1-\nu)/2)}(k\alpha^2)^{\nu/2}\exp\left\{\frac{i\pi\nu}{4} - \frac{ik\alpha^2}{2}\right\}\int_0^\infty \exp\left\{\frac{ix^2}{4k\alpha^2} - x\right\}x^{-\nu-1}\,dx \quad (\operatorname{Re}\nu < 0). \quad \text{(A-7)}$$

APPENDIX B. THE DISTRIBUTION OF THE ROOTS OF B

In order to locate the roots of the equation

$$B_\nu^{(1)}(\beta) = 0 \quad (\nu = \sigma + i\tau), \quad \text{(B-1)}$$

we make use of the relation

$$\overline{B}_\nu^{(1)}(\beta)\, B_\nu^{(1)\prime}(\beta) - \overline{B}_\nu^{(1)\prime}(\beta)\, B_\nu^{(1)}(\beta) = 2ik(1+2\sigma)\int_0^\beta |B_\nu^{(1)}|^2\, d\beta,$$

**Here the upper sign refers to the region $\operatorname{Re}\{\nu\} < -1/2$, the lower to the region $\operatorname{Re}\{\nu\} > -1/2$.

***This formula was obtained from the well-known integral representation[10]

$$F(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{tz}\,dt; \quad (\operatorname{Re} b > \operatorname{Re} a > 0)$$

by the change of variables $(1-t)/t = e^{2\sigma}$,

[10] N. N. Lebedev, *Special Functions and their application*, GTTI, 1953.

which is easily obtained from the differential equations and initial conditions for the functions under consideration. If we assume that, for given β, the value of ν is such that $\beta_\nu^{(1)}(\beta)=0$ then $\beta_{\bar{\nu}}^{(1)}(\beta)=0$ also, and consequently, Re $\{\nu\}=\sigma=-1/2$, i.e., all the roots ν_n of the function $\beta_\nu^{(1)}$ $\beta_\nu^{(1)}(\beta)$ are located to the right of Re $\{\nu\}=-1/2$.

As a more detailed investigation shows, the imaginary parts of these roots take the following asymptotic forms for $|\tau_n|\to\infty$:

$$\tau_n \approx \frac{1}{2k\rho^2}\left(\frac{2n-1}{2}\pi\right)^2. \tag{B-2}$$

One can also show that the number τ_n is bounded below.

APPENDIX C. DERIVATION OF AN ESTIMATE FOR THE FUNCTION

The purpose of this appendix – to establish some estimate for functions of the third type $A_\nu^{(3)}(\alpha)$, uniformly valid in α for $-\tau < Im\{\nu\} < \infty$ (T an arbitrarily large positive number).

Integrating Eq. (A-7) by parts, we find

$$A_\nu^{(3)}(\alpha) = \frac{2^{\nu+1}\sqrt{\pi}}{\Gamma(1-\nu/2)}(k\alpha^2)^{\nu/2}\exp\left\{\frac{i\pi\nu}{4}-\frac{ik\alpha^2}{2}\right\}$$
$$\times\frac{1}{\nu}\int_0^\infty x^{-\nu}\exp\left\{\frac{ix^2}{4\alpha^2 k}-x\right\}\left(\frac{ix}{2k\alpha^2}-1\right)dx;$$
$$\nu=-\delta+i\tau;\quad \delta>0.$$

Estimating the modulus of the latter integral, and taking into account the behavior of gamma functions for large τ, we get

$$|A_\nu^{(3)}(\alpha)| = \alpha^{-\delta}\varphi(\tau)\quad(\delta>0), \tag{C-1}$$

where

$$\varphi(\tau)=\begin{cases}O(1) & \text{for } -T\leqslant\tau\leqslant T,\\ O(\tau^{-1-\delta/2}) & \text{for } T<\tau<\infty.\end{cases} \tag{C-2}$$

Making use further of the integral representations for the linear combination $\alpha^{-1}A_\nu^{(3)\prime}(\alpha)+ikA_\nu^{(3)}(\alpha)$ [easily obtained from Eq. (A-7)], we find, as above,

$$\left|\frac{1}{\alpha}A_\nu^{(3)\prime}(\alpha)+ikA_\nu^{(3)}(\alpha)\right| = \alpha^{-2-\delta}\psi(\tau), \tag{C-3}$$

where

$$\psi(\tau)=\begin{cases}O(1) & \text{for } -T\leqslant\tau\leqslant T,\\ O(\tau^{-1-\delta/2}) & \text{for } T<\tau<\infty.\end{cases} \tag{C-4}$$

Translated by R. T. Beyer
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