Meson Field Theory. III. Conservation of Physical Quantities

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The paper investigates densities of physical quantities and conservation laws for meson fields with anticommuting matrices, whose general properties were described in previously published papers[1-3]

CONSERVATION laws for wave fields can be obtained using Noether's theorem.[4,5] This method is based on the use of the correspondence which exists between the invariance of the Lagrangian of the field with respect to coordinate transformations of the conformal group and the conservation of physical quantities.

In the present case, such a procedure is not convenient mainly because of the non-uniqueness of the variational principle. In obtaining the conservation laws it is therefore more expedient to go directly to the wave equations, since any conservation laws must be consistent with them. We choose the wave equations of the meson field in the form

$$\Gamma_\lambda \partial\Psi / \partial x_\lambda + k_0\Psi = Q, \tag{1}$$

$$-\partial\Psi^+ / \partial x_\lambda \Gamma_\lambda + k_0\Psi^+ = Q^+,$$

where the kinematic matrices Γ_μ anticommute:

$$\tfrac{1}{2}\{\Gamma_\mu\Gamma_\nu\} - \delta_{\mu\nu}\cdot I = 0. \tag{2}$$

If we use a representation with real Γ_i and imaginary Γ_4, we can define the charge conjugate by means of the diagonal reflection matrix R_4

$$\Psi^+ = \Psi^* R_4 \qquad (R_\mu = 2\beta_\mu^2 - I). \tag{3}$$

The anticommuting matrices Γ_μ, which, together with their products, form the group G_{16}, are made up of two orthogonal 16'th-degree reducible representations of the Kemmer algebra:

$$\Gamma_\mu = \beta_\mu^{(1)} + \beta_\mu^{(2)}, \tag{4}$$

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu - \beta_\mu\delta_{\nu\rho} - \beta_\rho\delta_{\nu\mu} = 0, \tag{5}$$

$$\beta_\mu^{(1)}\beta_\mu^{(2)} = 0 \text{ (n. s.)*}$$

The orthogonalization can be achieved most simply by using the matrix of the genalized Larmor transformation[2]:

$$\beta_\mu^{(2)} = -\Gamma_5\beta_\mu^{(1)}\Gamma_5. \tag{6}$$

where $(\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4)$:

In addition to the group G_{16}, another group $\bar{G}_{16}$ generated by the anticommuting matrices $\bar{\Gamma}_\mu$

$$\bar{\Gamma}_\mu = \beta_\mu^{(1)} - \beta_\mu^{(2)}, \tag{7}$$

$$\tfrac{1}{2}\{\bar{\Gamma}_\mu\bar{\Gamma}_\nu\} - \delta_{\mu\nu}\cdot I = 0, \quad [\Gamma_\mu\bar{\Gamma}_\nu] = 0. \tag{8}$$

will play an important part.

It is easy to see that all elements of G_{16} and $\bar{G}_{16}$ commute with one another. The matrices Γ_μ and $\bar{\Gamma}_\mu$ are related simply via the reflection matrices $:\bar{\Gamma}_\mu = R_\mu\Gamma_\mu$ (n. s.)

From the field equations, we immediately obtain differential relations which are bilinear in the components of the undor** Ψ

$$\Psi^+ M\Gamma_\lambda\partial\Psi / \partial x_\lambda \tag{9}$$

$$\mp(\partial\Psi^+ / \partial x_\lambda)\Gamma_\lambda M\Psi + (1 \pm 1)k_0\Psi^+ M\Psi = \Psi^+ MQ \pm Q^+ M\Psi,$$

where M is any matrix of G_{16} or $\bar{G}_{16}$.

Conservation laws are obtained from (9) in the

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[1] A. A. Borgardt, Dokl. Akad. Nauk SSSR **78,** 1113 (1951).

[2] A. A. Borgardt, J. Exptl. Theoret. Phys. (U.S.S.R.) **24,** 24 (1953).

[3] A. A. Borgardt, J. Exptl. Theoret .Phys. (U.S.S.R.) **24,** 284 (1953).

[4] E. Bessel-Hagen, Math. Ann. **84,** 258 (1921).

[5] E. L. Hill, Revs. Mod .Phys. **23,** 253 (1951).

*n. s. means 'no summation.'

**By undor we designate a one-column matrix whose 16 elements consist of a scalar, a 4-vector, and the components of completely antisymmetric tensors of all ranks up to four.

two cases where M commutes, or anticommutes, with all the Γ_μ.

a). $[M\Gamma_\mu] = 0$. This can happen if M belongs to the ring $\overline{G}_{16}$; we get 16 conservation laws of the form

$$\partial(\Psi^+\overline{\Gamma}\Gamma_\lambda\Psi)/\partial x_\lambda = \Psi^+\overline{\Gamma}Q - Q^+\overline{\Gamma}\Psi, \tag{10}$$

where $\overline{\Gamma}$ is any element of $\overline{G}_{16}$.

The conserved quantities have the following physical meaning:

1. $\overline{\Gamma} = I$, $j_\mu = (e/2\mathcal{E}_0)\Psi^+\Gamma_\mu\Psi$

is the vector of the electric current density, and vanishes in a neutral field (the proof was given in Ref. 3, Eqs. 25-27; the problem is handled similarly in all other cases.)

2. $\overline{\Gamma} = \overline{\Gamma}_\mu$, $T'_{\mu\nu} = {}^1/_2\Psi^+\overline{\Gamma}_\mu\Gamma_\nu\Psi$

the symmetric energy-momentum tensor, does not vanish in a neutral field.

3. $\overline{\Gamma} = {}^1/_2[\overline{\Gamma}_\mu\overline{\Gamma}_\nu]$, $\sigma'_{\mu\nu,\rho} = (c/4\mathcal{E}_0)\Psi^+[\overline{\Gamma}_\mu\overline{\Gamma}_\nu]\Gamma_\rho\Psi$,

the spin pseudovector, does not vanish in a neutral field. The form of the conservation equation is

$$\partial\sigma'_{\mu\nu,\lambda}/\partial x_\lambda = (c/4\mathcal{E}_0)(\Psi^+[\overline{\Gamma}_\mu\overline{\Gamma}_\nu]Q - Q^+[\overline{\Gamma}_\mu\overline{\Gamma}_\nu]\Psi) \tag{11}$$

and is expressed in three-dimensional form as

$$\partial\vec{\sigma}/c\partial t + \nabla, \ldots \tag{12}$$

Thus, strictly speaking, we are dealing simply with the mechanical equation of motion of an angular momentum.

The space divergence which appears on the left side of (12) represents the time derivative of the orbital angular momentum. Therefore we should distinguish (11) from the strict spin-conservation equation (continuity equation) which will be gotten later.

4. $\overline{\Gamma} = \overline{\Gamma}_5\overline{\Gamma}_\mu$, $M_{\rho\tau} = (e\hbar c/2\mathcal{E}_0^2)\Psi^+\overline{\Gamma}_5\overline{\Gamma}_\mu\Gamma_\nu\Psi$,

the antisymmetric tensor of the dipole moment, vanishes in a neutral field.

5. $\overline{\Gamma} = \Gamma_5$, $\sigma''_\mu = (ic/2\mathcal{E}_0)\Psi^+\overline{\Gamma}_5\Gamma_\mu\Psi$,

a spin pseudovector, vanishes in a neutral field since, like the current and the dipole moment, it is a quantity of the type ( ) – ( )*. Obviously, the spin σ''_μ (and not σ'_μ) is responsible for the formation of the kinematic dipole moment $M_{\mu\nu}$. In addition, σ'_μ and σ''_μ differ in the sign of terms containing ψ_0 and $\overline{\psi}_0$; in a purely transverse field, this difference disappears. The spin σ''_μ satisfies an equation of continuity of the form

$$\partial(\Psi^+\overline{\Gamma}_5\Gamma_\lambda\Psi)/\partial x_\lambda = \Psi^+\overline{\Gamma}_5 Q - Q^+\overline{\Gamma}_5\Psi. \tag{13}$$

b). $\{M\Gamma_\mu\} = 0$. This can occur if M belongs to $R_5\overline{G}_{16}$; we get 16 conservation laws of the form

$$\partial(\Psi^+R_5\Gamma\Gamma_\lambda\Psi)/\partial x_\lambda + 2k_0\Psi^+R_5\overline{\Gamma}\Psi = \Psi^+R_5\overline{\Gamma}Q + Q^+R_5\overline{\Gamma}\Psi. \tag{14}$$

The conserved quantities differ from those considered in case a) by the presence of the matrix R_5 of reflection of proper time. Its effect on the density can be made clear from the following considerations. The matrix R_5 is used for splitting the wave function Ψ into a potential matrix Ψ^{I} and a field strength matrix Ψ^{II}[3]:

$$\Psi^{\mathrm{I}} = ({}^1/_2 k_0)(I - R_5)\Psi, \qquad \Psi^{\mathrm{II}} = {}^1/_2(I + R_5)\Psi. \tag{15}$$

Since $R_5^2 = I$, we get

$$R_5\Psi^{\mathrm{I}} = -\Psi^{\mathrm{I}}, \quad R_5\Psi^{\mathrm{II}} = \Psi^{\mathrm{II}}. \tag{16}$$

Densities of physical quantities can be subdivided into those of "symmetrical" type

$$\rho_s(\Gamma) = \Psi^{+\mathrm{II}}\Gamma\Psi^{\mathrm{II}} + k_0^2\Psi^{+\mathrm{I}}\Gamma\Psi^{\mathrm{I}} = \rho^{\mathrm{II}} + k_0^2\rho^{\mathrm{I}} \tag{17}$$

and those of "antisymmetrical" type

$$\rho_{as}(\Gamma) = k_0(\Psi^{+\mathrm{II}}\Gamma\Psi^{\mathrm{I}} + \Psi^{+\mathrm{I}}\Gamma\Psi^{\mathrm{II}}). \tag{18}$$

The introduction of R_5, while it does not change the type of quantity, converts an antisymmetric density of the form ( ) + ( )* into a density of the form ( ) – ( )*, and vice versa; in the symmetric case, the sign of the part $\sim k_0^2$ is reversed.

The conserved quantities are:

1. $\overline{\Gamma} = I$, $s_\mu = (c/2\mathcal{E}_0)\Psi^+R_5\Gamma_\mu\Psi$,

the action current vector, does not vanish in a neutral field.

2. $\overline{\Gamma} = \overline{\Gamma}_\mu$, $T''_{\mu\nu} = {}^1/_2\Psi^+R_5\overline{\Gamma}_\mu\Gamma_\nu\Psi$

a symmetric energy-momentum tensor, does not vanish in a neutral field; it differs from $T'_{\mu\nu}$ in the sign of the part $\sim k_0^2$.

3. $\overline{\Gamma} = {}^1/_2[\Gamma_\mu\Gamma_\nu]$, $\overline{\sigma}'_{\mu\nu,\rho}(c/4\mathcal{E}_0)\Psi^+R_5[\overline{\Gamma}_\mu\overline{\Gamma}_\nu]\Gamma_\rho\Psi$,

a spin pseudovector, differs from $\sigma'_{\mu\nu,\rho}$ in the fact that it vanishes in a neutral field.

4. $\overline{\Gamma} = \overline{\Gamma}_5\overline{\Gamma}_\mu$, $\overline{M}_{\rho\tau} = (e\hbar c/2\mathcal{E}_0^2)\Psi^+\Gamma_5\overline{\Gamma}_\mu\Gamma_\nu\Psi$,

an antisymmetric dipole moment tensor, differs from $M_{\rho\tau}$ in the sign of the part $\sim k_0^2$.

5. $\overline{\Gamma} = \overline{\Gamma}_5$, $\overline{\sigma}''_\mu = (ic/2\mathcal{E}_0)\Psi^+\Gamma_5\Gamma_\mu\Psi$

a spin pseudovector, differs from σ''_μ in the fact that it does not vanish for a neutral field.

In a purely transverse wave field, the equalities $\sigma'_\mu = \bar{\sigma}''_\mu$ and $\sigma''_\mu = \bar{\sigma}'_\mu$. hold. Thus the wave field possesses two spin densities, for each of which there is an equation of motion and an equation of continuity.

The use of two sets of anticommuting matrices Γ_μ and $\bar{\Gamma}_\mu$ also simplifies the theory of ordinary meson fields. From (4) and (7), it follows immediately that the equations

$$^1/_2 (\Gamma_\lambda + \bar{\Gamma}_\lambda)\, \partial\Psi / \partial x_\lambda + k_0 \Psi = {}^1/_2 (Q + \bar{Q}), \quad (19)$$

$$- {}^1/_2 (\Gamma_\lambda + \bar{\Gamma}_\lambda)^T \partial\Psi^+ / \partial x_\lambda + k_0 \Psi^+ = {}^1/_2 (Q + \bar{Q})^+$$

describe the usual vector-pseudoscalar field. Current, spin, energy, momentum and other quantities are formed from those considered above by symmetrizing with respect to Γ_μ and $\bar{\Gamma}_\mu$, at the same time preserving the parity of the quantity with respect to coordinate indices.

$$j_\mu = (e/4\mathcal{E}_0)\, \Psi^+ (\Gamma_\mu + \bar{\Gamma}_\mu)\, \Psi = (e/2\mathcal{E}_0)\, \Psi^+ \beta_\mu \Psi, \quad (20)$$

$$\sigma_\mu = (ic/4\mathcal{E}_0)\, \Psi^+ (\Gamma_5 \Gamma_\mu + \bar{\Gamma}_5 \bar{\Gamma}_\mu)\, \Psi$$

$$= (ic/2\mathcal{E}_0)\, \Psi^+ \beta_5 \beta_\mu \Psi,$$

$$T_{\mu\nu} = {}^1/_2\, \Psi^+ (\bar{\Gamma}_\mu \Gamma_\nu + \Gamma_\mu \bar{\Gamma}_\nu)\, \Psi$$

$$= {}^1/_2\, \Psi^+ (\{\beta_\mu \beta_\nu\} - 2\delta_{\mu\nu} \cdot I)\, \Psi$$

etc. These same quantities for a pseudovector-scalar field are constructed by applying a generalized Larmor transformation to the operator of the corresponding quantity; this leaves the energy-momentum tensor unchanged.

In conclusion we must point out the following: In contrast to the equations obtained using Noether's theorem, the conservation equations given above contain explicitly only internal field quantities. However, it is easy to verify that this difference is a purely formal one.

We shall consider the conservation of the action current for a free electromagnetic field, and use vector notation. In this case, the quantity $s_\mu = (s, is_0)$,

$$\mathbf{s} = (1/c)\, [\mathbf{H}\mathbf{A}], \quad s_0 = -(1/c)\, \mathbf{E}\mathbf{A}, \quad (21)$$

satisfies a conservation law of the form of (14):

$$\partial s_\lambda / \partial x_\lambda = \partial s_0 / c\partial t + \nabla \mathbf{s} = {}^1/_2 c\, (\mathbf{E}^2 - \mathbf{H}^2). \quad (22)$$

On the other hand, using Noether's theorem, the invariance of the Lagrangian under reflection, $\delta x_\mu = a x_\mu$ leads to the conservation law

$$(\partial / \partial x_\lambda)\, (F_{\lambda\sigma} A_\sigma + x_\sigma T^{(k)}_{\lambda\sigma}) \quad (23)$$

$$= (\partial / c\partial t)\, (\mathbf{E}\mathbf{A}/c - \mathcal{E}^{(k)} t + \mathbf{p}^{(k)} \mathbf{x})$$

$$+ \nabla \cdot ([\mathbf{H}\mathbf{A}]/c - \mathbf{S}^{(k)} t + \mathbf{T}^{(k)} x) = 0,$$

where $\mathbf{p}^{(k)}$, $\mathbf{S}^{(k)}$, $\mathcal{E}^{(k)}$, $T^{(k)}_{ik}$, the momentum density, energy flux, energy density and stress tensor are components of the canonical energy-momentum tensor $T^{(k)}_{\mu\nu}$

In order to show the equivalence of (22) and (23), we substitute in (23) the quantity $\partial s_0 / c\partial t + \nabla \cdot \mathbf{S}$ from (22), and integrate over the four-dimensional volume occupied by the field. We then obtain the obvious relation between the action, energy, and momentum of the field:

$$S = -t \int \mathcal{E}^{(k)} (d\mathbf{x}) + \int \mathbf{x} \mathbf{p}^{(k)} (d\mathbf{x}). \quad (24)$$

A similar equivalence proof can also be given in other cases.

Translated by M. Hamermesh
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