It is obvious from the above that in the initial phase of the process diffusion plays a secondary role, but asymptotically, for large values of t, the speed of growth is determined exclusively by diffusion. Actually it is seen from Eq. (9) that when $t \to \infty$ $V \approx at^{-1/2}$; a is determined by substitution into (9), and we obtain the asymptotic speed of growth, which coincides with (7).

Our conclusions, as one can easily be convinced, are preserved for a different choice of conditions on the face, for instance, for a condition of the type

$$V(t) = k\,[c(0,\,t) - c_0]^n. \tag{11}$$

Equation (9) can be easily solved for the case of a process of the first order, $n = 1$. Dividing Eq. (9) for $n = 1$ by $\sqrt{t'' - t}$, , integrating with respect to t from 0 to t'', and changing the order of integration in the last term, we get the equation

$$\frac{\sqrt{\pi D}}{\delta}(c_\infty - c_0) - \frac{\sqrt{\pi D}}{k\delta}V(t) = 2k(c_\infty - c_0)\sqrt{t} - \frac{k\delta\sqrt{\pi}}{\sqrt{D}}\int_0^t V(t')\,dt',$$

which has the following solution

$$V(t) = k(c_\infty - c_0)\,e^{k^2\delta^2 t/D}\operatorname{erfc}(k\delta\sqrt{t/D}). \tag{12}$$

Using the asymptotic expansion

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\,e^{-z^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{n!\,(2z)^{2n-1}},$$

we obtain the known expression for the asymptotic behavior of the speed of growth and the criterion for the applicability of diffusion theory in the form

$$k^2\delta^2 t/D \gg 1. \tag{13}$$

From Eq. (8) one can easily determine the thickness of the depleted zone: $\Delta = 2\sqrt{Dt}$. Then criterion (13) assumes the form $k\delta\Delta/D \gg 1$. Diffusion is the dominating factor after the solution is depleted in an adequately wide zone; however, this factor is inhibiting. Mixing and convection reduce the influence of diffusion to zero, and the properties of the surface become of primary importance. The role of heat-transfer during crystallization after smelting is considered in a similar way.

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Relativistic Repulsion Effects in a Scalar Field and Attraction Effects in a Vector Field

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IN the recent times, it has been shown by a number of authors[1-6] that in the relativistic theory of a particle moving in a scalar field, the effect of relativistic repulsion takes place. For instance, in the case of a radial field with a purely attractive potential this effect produces a repulsion in the neighborhood of the center. In quantum mechanics this effect has been discovered and investigated by Kuni and Taksar[1], in the case of a Dirac particle with spin. The effect of relativistic repulsion for a classical particle has been investigated most prominently by Werle[2], Marks and Chamosi[3] and also by Infeld.[4] Werle[5] considered also the case of a spinless quantum particle, subject to the Klein-Gordon equation, in a scalar and a vector field, and showed that, in the non-relativistic approximation, some additional effective potentials appear which can be considered as repulsive potentials in the case of a scalar field, and as attractive potentials in the case of a static vector field. It is easy to show, however, that the latter deduction can be made without making a non-relativistic approximation. Furthermore, for a spinless particle, the existence of relativistic effects--repulsive in a scalar field and attractive in a static vector field--can be deduced in a very convincing and clear manner.

The Klein-Gordon equation for a spinless particle in a scalar field with potential Φ, and simultaneously in a vector field with potential A, A_4

has the form:

$$\left[\left(\frac{\hbar}{i}\nabla - \mathbf{A}\right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - A_4\right)^2\right]\psi = -\frac{1}{c^2}[mc^2 + \Phi]^2\psi. \quad (1)$$

In the case of a static vector potential with components A=0, $A_4 = U$, assuming for the static case, as usual, $\psi = u \exp(-iEt/h)$ we get:

$$\nabla^2 u + \hbar^{-2}c^{-2}[(E-U)^2 - (mc^2+\Phi)^2]u = 0. \quad (2)$$

The latter equation can easily be written in a form identical to the nonrelativistic stationary Schrödinger equation. Introducing the notation

$$U' = \left(\Phi + \frac{\Phi^2}{2mc^2}\right) + \left(\frac{E}{mc^2}U - \frac{U^2}{2mc^2}\right), \quad (3)$$

$$E' = (E - mc^2)\left(1 + \frac{E - mc^2}{2mc^2}\right),$$

Eq. (2) can be written in the form:

$$-(\hbar^2/2m)\nabla^2 u + [U' - E']u = 0. \quad (4)$$

In this manner, the eigenfunctions of the considered problem may be found as eigenfunctions of an equivalent nonrelativistic problem with effective potential U'. In the case of a purely scalar field, i.e., for $U=0$, Eq. (4) is identical to the usual stationary Schrödinger equation with a potential $U'=\Phi+\Phi^2/2mc^2$. In the case of a pure vector static field, i.e., for $\Phi=0$, Eq. (4) must be considered as some generalization of the Schrödinger equation, as far as the parameter E is involved in the expression of the effective potential $U' = (EU - \frac{1}{2}U^2)/mc^2$. However, for each given value of E, the function U' can be considered as an effective potential. It is with this potential, in particular, that the definite solution of Eq. (2) with $E=E_k$ is determined.

One sees from the expression (3), that, in the case of a potential monotonically decreasing towards the center, i.e., in the case of an attractive potential $\Phi(r)$, the effective potential $U'(r)$ increases towards the center, i.e., becomes repulsive for $r<r_0$, where r_0 is determined by $\Phi=(r_0)=-mc^2$. In the case of a potential, monotonically increasing towards the center, i.e., in the case of a repulsive potential $U(r)$, the effective potential $U'(r)$ becomes decreasing towards the center, i.e., becomes attractive for $r<r_m$ where r_m is determined by $U(r_m)=E$. However, for a repulsive potential $\Phi(r)$ of a scalar field, the effective potential is everywhere repulsive, and for an attractive potential $U(r)$ of a vector field, the effective potential is everywhere attractive.

The effect of relativistic repulsion in a scalar field has been satisfactory studied in the literture.[1-4] To take the simplest example of relativistic attraction effect in a repulsive vector field, let us consider the problem of the motion of a scalar particle in a static field with a potential which is a step function, high at the origin, i.e., a potential:

$$\Phi = 0, \quad U(r) = \begin{cases} U_m & \text{for } r < r_0, \\ 0 & \text{for } r > r_0, \end{cases} \quad U_m > 0. \quad (5)$$

For the function $\chi(r)=ru(r)$ in the state $l=0$, Eq. (2) takes the form

$$(d^2\chi/dr^2) + \hbar^{-2}c^{-2}[(E-U)^2 - m^2c^4]\chi = 0. \quad (6)$$

It is easy to see, that for sufficiently large values of U_m there exist solutions of (6) which are of the form

$$\chi(r) = \begin{cases} A\sin k_1 r & \text{for } r < r_0, \\ Be^{-k_2 r} & \text{for } r > r_0, \end{cases} \quad E < mc^2 \quad (7)$$

$$k_1^2 = (\hbar c)^{-2}[(E-U_m)^2 - m^2c^4], \quad k_2^2 = (\hbar c)^{-2}(m^2c^4 - E^2),$$

From the continuity conditions on the function χ and on its derivatives at $r=r_0$ it follows that the coefficients k_1 and k_2 are subject to the relationship $\tan r_0 k_1 = -k_1/k_2$. The eigenvalues E are determined from this equation. It is easy to see that it has solutions with eigenvalues E lying in the interval $-mc^2 < E < mc^2$ for $U_m > E + mc^2$ (then $k_1 > 0$, $k_2 > 0$).

The easiest way to convince oneself in the truth of the last statement is to take a constant E in the interval mentioned above (for instance take $E=0$), and to vary U_m, i.e., take a constant k_2 and vary k_1.

Hence, for a purely repulsive potential (7) there are solutions of Eq. (2) for $E < mc^2$, which have the same form as the solutions of a nonrelativistic problem with an attractive step-function potential. However, in the case of a pure Coulomb repulsive potential $U=e^2/r$, the relativistic attractive effect in a vector field does not give rise to stable or metastable energy levels. Indeed, in this case, the maximum value of the effective potential is, according to (3): $U'_{max} = E^2/2mc^2$ and the minimum possible value of E' for a particle localized in the region of "attraction" can be estimated, using the uncertainty relations, to be $(\hbar c/e^2)^2\, 3E^2/2mc^2$. Then the ratio $E'/U'_{max} \sim 3(\hbar c/e^2)^2 = 3(137)^2$, i.e, the lowest level E' lies much higher than U'_{max}.

It is clear that the relativistic attraction effect in a quasi-coulombic repulsive potential with potential g^2/r can give rise to metastable levels only if the constant g is more than 15 times larger than the elementary charge e.

The calculations shown above confirm that in the relativistic quantum theory for a spinless particle in a scalar purely attractive field, appears a repulsion effect, and in a static vector, purely repulsive field, an attractive effect takes place.

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The Kinetics of Paramagnetic Phenomena

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IN the kinetic theory of electrons in metals, it usually suffices to limit oneself to the use of the distribution function $f(\mathbf{p},\mathbf{r})$ which gives the number of electrons per cell of phase space. However, for processes related to the change of spin states of the electrons, this appears to be insufficient and one has to introduce the vector phase space magnetization density $\vec{\sigma}(\mathbf{p},\mathbf{r})$, which is a generalization of the density matrix in the mixed representation for the case of a system of particles with spin ½. One can write the following equation* for $\vec{\sigma}(\mathbf{p},\mathbf{r})$:

$$\frac{\partial\vec{\sigma}}{\partial t}+\left(\frac{\mathbf{p}\partial}{m\partial\mathbf{r}}\right)\vec{\sigma}+e\left(\mathbf{E}+\frac{[\mathbf{pH}]}{mc},\frac{\partial}{\partial\mathbf{p}}\right)\vec{\sigma}+\frac{\beta}{\hbar}[\vec{\sigma}\mathbf{H}]=-\mathbf{J}_U-\mathbf{J}_\tau \tag{1}$$

where $\beta/2$ is the effective magnetic moment of the particle and $\mathbf{J}_\tau$ and $\mathbf{J}_U$ are integral operators, taking into account collisions without and with spin change, respectively. Usually the corresponding relaxation times τ and U satisfy the inequality $U \gg \tau$, hence the introduced separation between the integral operators does not give rise to any problems. Note that $\mathbf{J}_\tau$ and $\mathbf{J}_U$, generally speaking, depend on $\vec{\sigma}$ as well as on f. We shall not write their exact expression. We can approximate $\mathbf{J}_U$ by:

$$\mathbf{J}_U=U^{-1}(\vec{\sigma}-\vec{\sigma}_{00}), \tag{2}$$

where U is the spin relaxation time, and $\vec{\sigma}_{00}$ is the equilibrium value of the phase density of magnetization, which differs from zero in the case of a permanent magnetic field. In general, one cannot use for $\mathbf{J}_\tau$ an approximation similar to (2), because

$$\int \mathbf{J}_\tau d\mathbf{p}=0, \tag{3}$$

which is inconsistent with the equation analogous to (2). Equation (3) is an obvious consequence of the fact that a collision without spin change does not change the magnetization.

Being interested in the equation for the space magnetization density $\mathbf{M}(\mathbf{r},\mathbf{t})$ we assume that

$$\vec{\sigma}=\vec{\sigma}_0+\vec{\Sigma}=\mathbf{M}(\mathbf{r},t)F(p^2)+\vec{\Sigma}, \tag{4}$$

where

$$\int d\mathbf{p}F(p^2)=1,\quad \int d\mathbf{p}\vec{\Sigma}=0. \tag{5}$$

Then, from (1), we get the following system of equations:

$$\frac{\partial\mathbf{M}}{\partial t}+\int d\mathbf{p}\left(\frac{\mathbf{p}}{m}\frac{\partial}{\partial\mathbf{r}}\right)\vec{\Sigma}+\frac{\beta}{\hbar}[\mathbf{MxH}]=-\frac{1}{U}(\mathbf{M}-\mathbf{M}_0), \tag{6}$$

$$F(p^2)\left(\frac{\mathbf{p}}{m}\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{M}+\mathbf{M}e\left(\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)F+\frac{\partial\vec{\Sigma}}{\partial t}+\frac{\vec{\Sigma}}{U}+\left(\frac{\mathbf{p}}{m}\frac{\partial}{\partial\mathbf{r}}\right)\vec{\Sigma}-F\int d\mathbf{p}'\left(\frac{\mathbf{p}'}{m}\frac{\partial}{\partial\mathbf{r}}\right)\vec{\Sigma}(\mathbf{p}')+e\left(\mathbf{E}+\frac{[\mathbf{pxH}]}{mc},\frac{\partial}{\partial\mathbf{p}}\right)\vec{\Sigma}+\frac{\beta}{\hbar}[\vec{\Sigma}\mathbf{xH}]=-\mathbf{J}_\tau(\vec{\sigma}_0+\vec{\Sigma}) \tag{7}$$

to get the equation for $\mathbf{M}$ it is sufficient, with the help of (7) to express $\vec{\Sigma}$ in terms of M. This is not difficult to do in the case $\Sigma \ll \sigma$ i.e., when the part