The Multiplicative Renormalization Group in the Quantum Theory of Fields

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We have derived Lee's differential equations for the multiplicative normalization group in the quantum theory of fields. As an example of the application of these equations, the Green's function of spinor electrodynamics has been derived in the regions of the infrared and ultraviolet catastrophes.

1. ONE of the basic problems of present day quantum theory of fields, and, in particular, of quantum electrodynamics, is the question of the behavior of the Green's function when large momenta are involved. It is well known that, in quantum electrodynamics, the expansions of the usual renormalized perturbation theory contain terms of the type $(e^2 \ln|k^2/m^2|)^n$. Due to these terms, the expansions can give the asymptotic behavior only in the region where $e^2 \ln|k^2/m^2| \ll 1$. There arises, therefore, the apparently complicated problem of summing the series of perturbation theories, in order to combine the important terms so as to put the results into the form

$$f(e^2 \ln|k^2/m^2|) + e^2 f_1(e^2 \ln|k^2/m^2|) + \dots, \quad (1)$$

In this, each of the coefficients $f, f_1, \dots f_n \dots$ represents the sum of an infinite number of Feynman diagrams of various orders.

The main term f of a series of the type (1) was found, for the basic Green's function of quantum electrodynamics, by Landau, Abrikosov and Khalatnikov[1] using asymptotic solutions of complicated integral equations which can be obtained by summing a definite class of "main" Feynman diagrams. However, it appears as if the problem of the transformation of the usual expansions of perturbation theory into the form (1), as well as a number of other problems (for example, that of understanding the singularity of the electronic Green's function in the "infrared region" $k^2 \sim m^2$) can be solved directly, without any summations of infinite series, using the renormalizing group of quantum electrodynamics. Attention was first called to the existence of such a group and to its general role in quantum field theory by Stückelberg and Petermann[2]. They also pointed out the possibility of introducing the corresponding infinitesimal operators, and of constructing the differential equations of Lee.

Starting with a different viewpoint, but in essence using the same renormalization group, Gell-Mann and Low[3] obtained concrete results on the behavior of Green's function in the asymptotic region of large momenta.

The purpose of this work is the extension of these ideas, and, in part, the derivation of general formulae for making use of the results of ordinary perturbation theory not only in the case of large momenta, but also in the region of the "infrared catastrophe."

2. In order to formulate clearly the equations describing the renormalization group of quantum electrodynamics, we recall briefly the main points of the procedure[4] for the removal of divergences from the scattering matrix, since the Green's functions can always be represented by this matrix with the help of variational derivatives. As usual, we will start by expanding the scattering matrix in powers of the interaction parameter. The coefficients in such an expansion are the chronological T-products of various numbers of Lagrangians of the interactions. Taking advantage of the arbitrariness implied in the T-products when all the arguments are equal, we can evaluate these products in such a way that taking into account the requirements of unitarity, causality, and covariance, the resulting operator functions are integrable. In order that intermediate steps should not involve actual infinities, we can introduce at this stage a formal regularization, for example, of the Pauli-

[1] L.D.Landau, A.A.Abrikosov, and I.M.Khalatnikov, Dokl. Akad. Nauk.SSSR. 95, 497,773, 1177 (1954); 96, 261 (1954).

[2] E.C.Stückelberg and A. Petermann, Helv. Phys. Acta. 26, 499 (1953).

[3] M.Gell-Mann and F.E.Low, Phys. Rev. 95, 1300 (1954).

[4] N.N.Bogoliubov and D.V.Shirkov, Usp. Fiz. Nauk 55, 149 (1955); 57, 3 (1955).

Villars type, which can be removed after the evaluation of the singularities. The recipe obtained for the evaluation of the T-products turns out to be equivalent to the usual subtraction formalism, which, in turn, is equivalent to the introduction into the original interaction Lagrangian of divergent counter terms. As a result of this procedure we get an expansion of the scattering matrix, no term of which contains divergencies. If it is assumed that an expansion of this type is summable, we now have to do with a finite scattering matrix and a finite Green's function. The finite expressions obtained in this way still contain an arbitrariness corresponding to the possibility of introducing (into the Lagrangian) finite counter terms of the same operator structure as the basic diverging counter terms. This arbitrariness, however, can likewise be removed if it be required that the theory describe particles with masses and charges equal to the experimentally known values of m_0 and e_0.

As has already been noticed by Dyson, the transition from the divergent scattering matrix to a finite one by use of subtraction techniques is equivalent (except for the counter term corresponding to intrinsic mass of the electron) to the multiplicative renormalization of the elementary Green's function and of the charge of the electron. A more detailed analysis of Dyson's discussions and their application to the general theory of Green's functions shows that, because of the presence of the counter term

$$\tfrac{1}{2}(Z_3-1)(\partial A_n/\partial x_n)^2$$

(which doesn't affect the scattering matrix elements because of the Lorentz conditions on acceptable states, and which is therefore usually thrown away), the subtraction procedure is equivalent to a multiplicative renormalization only when the unperturbed photon source function is chosen to be

$$D^{\mathrm{tr}}_{mn}(k)=\frac{i}{k^2}\left(g^{mn}-\frac{k_m k_n}{k^2}\right), \tag{2}$$

thus satisfying conditions of transversality.

This requirement was not noticed by Gell-Mann and Low, who tried to study the group equations for the Green's function of an electron by using the expression for the Green's function of a photon which is not transverse. As a result they arrived at a contradiction which we will discuss later.

As has already been mentioned, after removal of the infinities of the theory, there still remains a finite arbitrariness. We will now analyze in more detail this arbitrariness, paying special attention to its relation to the multiplicative renormalization of the Green's function and of the charge on the electron.

We will thus consider that, having removed the infinities by using some definite subtraction method, we have obtained finite Green's functions G and D, and the vertex part of Γ. In this case, due to the identity of the mass of the electron with its experimentally observed mass, the counterterm δm is uniquely determined, irrespective of any new multiplicative renormalization. The counterterm Z_3 is completely determined by the requirement that the electronic charge be the same as the experimentally measured e_0. At the same time, the multiplication of the diverging constant Z_3 by the finite factor z_3 leads to a finite renormalization of the charge

$$Z_3\to Z_3 z_3;\quad e\to z_3^{-1/2}e. \tag{3}$$

Finally, the counterterms Z_1 and Z_2 are found to be determined to within finite constants z_1 and z_2, subject only to the Ward identity

$$Z_1\to Z_1 z_1;\quad Z_2\to Z_2 z_2;\quad z_1=z_2. \tag{4}$$

In view of the fact that changing the counterterm in the investigated situation(2) leads to a new normalization of the Green's function, we arrive at the following set of finite transformation of finite quantities.

$$G_1\to G_2=z_2 G_1;\quad \Gamma_1\to\Gamma_2=z_1^{-1}\Gamma_1;\quad z_1=z_2; \tag{5}$$

$$D_1\to D_2=z_3 D_1;\quad e_1\to e_2=z_3^{-1/2}e_1.$$

The sense of the tranformations (5) is that the use of the quantities Γ_1, G_1, D_1 and e_1, in the theory, leads to the same results as the use of Γ_2, G_2, D_2 and e_2—that is, to a description of particles involving masses and charges equal to their experimental ones. The transformation group (5) represents the "renormalization group" of Stückelberg and Petermann[2], and makes it possible to get simple functional equations for the Green's functions that are similar to those of Gell-Mann and Low.[3]

3. Bearing in mind that such equations can be obtained, let us represent G and D in the form (the vertex part of Γ will be examined separately):

$$G(k)=i\,\frac{a(k^2)\,\hat{k}+b(k^2)\,m}{k^2-m^2}; \tag{6}$$

$$D_{mn}(k)=\frac{i}{k^2}\left(g^{mn}-\frac{k_m k_n}{k^2}\right)d(k^2). \tag{7}$$

It should be noted that the determination of G

and D involved two arbitrary constants z_2 and z_3 which can be determined by using the following relations:

$$d = 1, \quad a = 1 \quad \text{where } k^2 = \lambda^2, \tag{8}$$

when $\lambda^2 < 0$. The discussion that follows is easily generalized to the case of $\lambda^2 > 0$ where it suffices (due to the complex nature of the functions being studied when $k^2 > 0$) to study only their real parts separately. From considerations of single valuedness in momentum space it follows that a, b, and d have the form $f(k^2/\lambda^2, m^2/\lambda^2, e^2)$, wherefore, upon fixing the momentum λ,

$$a(1, m^2/\lambda^2, e^2) = d(1, m^2/\lambda^2, e^2) = 1. \tag{9}$$

It should be pointed out that the second of the conditions (8) does not appear to be necessary, since, if, for example, $k^2=\lambda^2$, $a=a_0$, then an unimportant multiplying factor will be involved in the following discussions of a_0.

The square of the momentum λ^2 can be directly related to the charge e. Insertion of (7) into the second equation of (5), and using the third equation of (5), we get

$$d(k^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2) = z_3\, d(k^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2), \tag{10}$$

where $e_2^2 = z_3^{-1} e_1^2$.

If we assume that $k^3=\lambda^2$, in (10), we get, from (9)

$$z_3 = d(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2) \tag{11}$$

from which there follows the functional equation for d

$$d\left(\frac{k^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right) = \frac{d(k^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2)}{d(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2)}, \tag{12}$$

in which

$$e_1^2 = e_2^2\, d(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2). \tag{13}$$

Let us now identify the charge e^2 occurring in Eqs. (12) and (13) with the observed values of the charge e_0^2. In this case the true photon function is normalized at the point $k^2=0$, and so has the form

$$d = d^0(k^2/m^2, e_0^2). \tag{14}$$

We have, consequently,

$$e^2 = e_0^2 d^0(\lambda^2/m^2, e_0^2). \tag{15}$$

Let us now treat the fermion Green's function. We will treat simultaneously the functions a and b, giving them for this purpose a general designation s. Having written down equations for s analogous to (10), and having determined the constant z_2, we get, after eliminating of z_2, the functional equation for s

$$s\left(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = s\left(1, \frac{m^2}{\lambda_2^2}, e_2^2\right)\frac{s(k^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)}{s(\lambda_2^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)} \tag{16}$$

4. We now proceed to the solution of the differential equations for d and s. If we make the substitutions $k^2/\lambda_2^2=x$, $m^2/\lambda_2^2=y$, $\lambda_1^2/\lambda_2^2=t$, we can write (12) and (16) in the form

$$e^2 d(x, y, e^2) = e^2 d(t, y, e^2)\, d(x/t, y/t, e^2 d(t, y, e^2)) \tag{17}$$

and

$$\ln s(x, y, e^2) = \ln s(x/t, y/t, e^2 d(t, y, e^2)) + \ln s(t, y, e^2) - \ln s(1, y/t, e^2 d(t, y, e^2)). \tag{18}$$

If we differentiate (17) and (18) with respect to x and then let $t=x$, we get the sought-for Lee equations in the form

$$\frac{\partial e^2 d(x, y, e^2)}{\partial x} = \frac{e^2 d(x, y, e^2)}{x} \times \left[\frac{\partial}{\partial \xi} d\left(\xi, \frac{y}{x}, e^2 d(x, y, e^2)\right)\right]_{\xi=1}; \tag{19}$$

$$\frac{\partial \ln s(x, y, e^2)}{\partial x} = \frac{1}{x} \times \left[\frac{\partial}{\partial \xi} \ln s\left(\xi, \frac{y}{x}, e^2 d(x, y, e^2)\right)\right]_{\xi=} . \tag{20}$$

We now see that to get the functions d and s for all values of their arguments it is sufficient to determine $d(k^2/\lambda^2, y, e^2)$ and $s(k^2/\lambda^2, y, e^2)$ only in the neighborhood of $k^2/\lambda^2 \sim 1$, for which one can use usual perturbation theory.

The equations obtained illustrate the fact that due to the renormalization group it is possible to vary the scale of momenta, at the same time changing the charge.

We now notice that actually we have only to solve equation (19), since, for a given d, the expression for s can be obtained from (20) by simple quadrature:

$$\frac{s(x, y, e^2)}{s(x_0, y, e^2)} = \int_{x_0}^{x} \frac{dz}{z}\left[\frac{\partial}{\partial \xi} \ln s\left(\xi, \frac{y}{z} e^2 d(z, y, e^2)\right)\right]_{\xi=1}. \quad (21)$$

The charge e^2 appearing here is related to the experimental charge e_0^2 by the expression

$$e^2 d(k^2/\lambda^2, m^2/\lambda^2, e^2) = e_0^2 d^0(k^2/m^2, e_0^2),$$

which makes it possible to consider e^2 as a function of λ.

We now make use of the general group equations that we have obtained, for the region of large impulses: $|m^2/k^2| \ll 1$. When we set $\lambda^2 \sim k^2$, we see that in the given region the expression

$$e_\lambda^2 d(k^2/\lambda^2, e_\lambda^2) \equiv e_\lambda^2 d(k^2/\lambda^2, 0, e_\lambda^2) \quad (22)$$

when $k^2 < 0$ asymptotically approaches $e_0^2 d^0(k^2/m^2, e_0^2)$. From the other side (22) is identically equal to $e_m^2 d(k^2/m^2, e_m^2)$. For this reason, letting d_{as}^0 indicate the asymptotic part of the function d^0, we have everywhere:

$$e_0^2 d_{as}^0(k^2/m^2, e_0^2) = e^2 d(k^2/m^2, e^2),$$

where $e^2 = e_m^2 = e_0^2 d_{as}^0(1, e_0^2)$. In exactly the same way, we can convince ourselves that the expression $s(k^2/m^2, e^2)$ is the asymptotic form of the functions to within an unimportant constant multiplier.

Thus, to get the asymptotic form of the functions in question we can insert into our equations (19), (21) $x = |k^2/m^2|$ and eliminate y. We then obtain:

$$\frac{\partial e^2 d(x, e^2)}{\partial x} = \frac{e^2 d(x, e^2)}{x} \varphi\{e^2 d(x, e^2)\}, \quad (23)$$

where

$$\varphi(e^2) = \left[\frac{\partial}{\partial \xi} d(\xi, e^2)\right]_{\xi=1} \quad (24)$$

and, moreover

$$\ln \frac{s(x, e^2)}{s(x_0, e^2)} = \int_{x_0}^{x} \frac{dz}{z}\left[\frac{\partial}{\partial \xi} \ln s(\xi, e^2 d)\right]_{\xi=1}. \quad (25)$$

Integrating the differential equation (23) by the usual method of separation of variables, we come to the equation obtained by Gell-Mann and Low.[3]

$$\int_{e^2}^{e^2 d} \frac{dz}{z\varphi(z)} = \ln x. \quad (26)$$

Furthermore, introducing into the integral (25) a new variable $e^2 d$ for z, we find on the basis of (23) an equation for the determination of s in the form

$$\ln \frac{s(x, e^2)}{s(x_0, e^2)} = \int_{e^2 d(x_0)}^{e^2 d(x)} \frac{dz}{z\varphi(z)}\left[\frac{\partial}{\partial \xi} \ln s(\xi, z)\right]_{\xi=1}. \quad (27)$$

We now make a series of observations about the Gell-Mann-Low equation in the form (26). First, it is obvious that in order to use this equation for the actual determination of d, it is necessary to have an expression for the function $\varphi(z)$. It is not hard to get such an expression with the help of perturbation theory.

We have, in fact

$$d^{-1}(\xi, e^2) = 1 - \frac{e^2}{3\pi} \ln \xi - \frac{e^4}{4\pi^2} \ln \xi + \ldots$$

and, therefore, on the basis of (24)

$$\frac{1}{\varphi(z)} = \frac{3\pi}{z}\left\{1 - \frac{3z}{4\pi} + a_1 z^2 + \ldots\right\}. \quad (28)$$

Next, it is not hard to conclude, from equation (26) that the magnitude of $e^2 d(x)$ cannot remain small for all values of x. Actually, for small z

$$1/\varphi(z) < (3\pi/z)(1 + c),$$

where c is a constant. For this reason, as long as $e^2 d$ is small, we will have:

$$\ln x < 3\pi(1 + c) \int_{e^2}^{e^2 d} \frac{dz}{z^2} < 3\pi(1 + c) \int_{e^2}^{\infty} \frac{dz}{z^2} = \frac{3\pi(1 + c)}{e^2}$$

and then the corresponding possible values of $|k^2|$ are bounded.

It should be pointed out that the argument of the function φ is the square of the charge. Because of this, in order to understand the behavior of the Green's function at extremely high momenta, when $e^2 d$ becomes of the order of unity and larger, it is necessary to examine the region of large charges (strong forces). In this connection it should be emphasized that the structure of $\varphi(z)$ here cannot be established on the basis of an analysis of a finite number of terms of an expansion of the type of (28). It would appear as if the region of such momenta is unimportant practically, since it is hardly to be expected that electrodynamics, as we know it, ignoring as it does heavy particles, is any good at energies of the order of $me^{137/2}$.

However, as is correctly pointed out by Gell-Mann and Low, the study of this region is an interesting mathematical problem, which might be useful in the construction of future theories. For example, if, as a result of more detailed investigations, it should turn out that

$$\int_{e^2}^{\infty} \frac{dz}{z\varphi(z)} < \infty,$$

then the hypothesis of Landau and Pomeranchuk concerning the essential incompleteness of present day electrodynamics would be confirmed. This follows since

$$\ln\left|\frac{k^2}{m^2}\right| < \int_{e^2}^{\infty} \frac{dz}{z\varphi(z)},$$

and the possible values of $|k^2|$ are therefore bounded, which is contrary to the assumption that the theory is local.

Since at present we do not have any other information about the behavior of the function $\varphi(z)$ than the expansion (28), it is necessary to restrict ourselves to momenta for which

$$(e^2/3\pi)\ln|k^2/m^2| < 1. \tag{29}$$

In this region a very simple expression for d can be obtained* from relations (26) and (28).

We emphasize that even the general problem of the construction of the renormalized usual expansion for d in the form (1) in the case where (29) holds can be solved directly. In fact, setting (28) into (26) we find

$$1 - d^{1} + \frac{3}{4\pi}e^2\ln d^{-1} + a_1e^4(d-1) + \ldots = \frac{e^2}{3\pi}\ln x,$$

from which we immediately get the sought-for expansion:

$$d^{-1}(x, e^2) = 1 - \frac{e^2}{3\pi}\ln x + \frac{3}{4\pi}e^2\ln\left(1 - \frac{e^2}{3\pi}\ln x\right) + e^4\ldots \tag{30}$$

Completely analogously we can get an expansion of the type (1) for the asymptotic form of the function $s = a, b$ which determines the electronic Green's function. For this it is only necessary to make use of our equation (27) and usual perturbation theory formulae:

$$a(x, e^2) = 1 + ce^4\ln x + \cdots \tag{31}$$

$$b(x, e^2) = b_1\left\{1 - \frac{3e^2}{4\pi}\ln x + \frac{e^4}{\pi^2}(\alpha_1\ln^2 x + \alpha_2\ln x) + \cdots\right\},$$

where c, b_1, α_1, α_2 are numerical coefficients.

We have:

$$\left[\frac{\partial}{\partial\xi}\ln a(\xi, z)\right]_{\xi=1} = cz^2 + \cdots, \left[\frac{\partial}{\partial\xi}\ln b(\xi, z)\right]_{\xi=1} = -\frac{3z}{4\pi} + \alpha_2\frac{z^2}{\pi^2} + \cdots$$

Inserting these expansions, together with (28) into equation (27), we will find that

$$\ln a(x, e^2) = e^2 3\pi c(d-1) + \cdots; \tag{32}$$

$$\ln\frac{b(x, e^2)}{b(1, e^2)} = \frac{9}{4}\ln d^{-1} + e^2\frac{3}{\pi}\left(\alpha_2 + \frac{9}{16}\right)(d-1) + \cdots$$

In order to get expressions of the type (1) for the functions a, b, it remains only to use d from equation (30).

If we restrict ourselves to the main terms, we get the formulae of Landau et al[1].

We emphasize that the formulae (31) have been obtained from perturbation theory using only the transverse source function of the photon (2). It is easy to see that if we use, instead of (2), the usual coupling

$$D^c_{mn} = ig^{mn}/k^2,$$

we arrive, in (32) at relations not agreeing with the true asymptotic behavior (1) for this situation. As we have mentioned, this contradiction was reached by Gell-Mann and Low who worked with the usual photon couplings.

As we see, the method used here does not require summation over infinite systems of Feynman diagrams. In order to determine the series (1) through members of order e^{2n} it is sufficient to have for the functions d, a, b, being investigated, only formulae of the usual renormalized perturbation theory to an accuracy of order e^{2n+2}. Their conversion

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* Comment in proof: Landau[5] has used this case to obtain corrections to the formula for d derived in Ref. 1.

[5] L. D. Landau, Nils Bohr and the Development of Physics, London, 1955, p. 52.

into form (1) is merely an algebraic operation.

All these remarks have been on the use of the general Lee equations of renormalized groups that we have obtained for the purpose of constructing the asymptotic parts of the Green's function for large impulses. We emphasize that the meaning of these equations is not at all limited by such a use.

For example, let us examine the region $k^2 \sim m^2$ in which the electronic Green's function has a singularity. Since d here is regular, we are interested only in the functions $s=a,b$. It will be convenient to express them in the form

$$s\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = S\left(\frac{k^2-m^2}{\lambda^2-m^2}, \frac{\lambda^2-m^2}{m^2}, e^2\right). \tag{33}$$

Let us now return to equation (21) and set $y=1$. Then, using the representation (33) we will get:

$$\ln\frac{s\,(k^2/m^2, e^2)}{s\,(k_0^2/m^2, e^2)} = \int\limits_{k_0^2/m^2}^{k^2/m^2} \frac{dx}{x-1}\left[\frac{\partial}{\partial\xi}\ln S(\xi, x-1, e^2 d(x, 1, e^2))\right]_{\xi=1}. \tag{34}$$

The interval of integration has to be taken as indicated in order not to fall into the pole $x-1=0$. For this reason we set $k^2/m^2>1$, $k_0^2/m^2>1$ or $k^2/m^2<1$, $k_0^2/m^2<1$. In order to use in practice, the relation (34) for the determination of s, we see that it is necessary to know the function $S(\xi,x-1,e^2)$ only in the infinitesimally small region of the point $\xi=(k^2-m^2)/(\lambda^2-m^2)=1$, which is its "point of normalization". For this reason, in order to obtain the part of (34) in square brackets, we again use perterbation theory formulae.

Taking the second approximation:

$$A(\xi, y, e^2) = 1 - \frac{3e^2}{2\pi}\left(\frac{\ln \xi y}{1+\xi y} - \frac{\ln y}{1+y}\right) + \ldots;$$

$$B(\xi, y, e^2) = 1 - \frac{3e^2}{2\pi}\left(\frac{2+\xi y}{1+2\xi y}\ln \xi y - \frac{\ln y}{1+y}\right) + \ldots;$$

$$e^2 d(x, 1, e^2) = e^2 + \ldots,$$

we find

$$\left[\frac{\partial}{\partial\xi}\ln A(\xi, x-1, e^2 d)\right]_{\xi=1} = \frac{3e^2}{2\pi}\left[\frac{(x-1)\ln|x-1|}{x^2} - \frac{1}{x}\right] + \ldots;$$

$$\left[\frac{\partial}{\partial\xi}\ln B(\xi, x-1, e^2 d)\right]_{\xi=1} = \frac{3e^2}{4\pi}\left[\frac{(x-1)\ln|x-1|}{x^2} - \frac{x+1}{x}\right] + \ldots,$$

and from (34) we get

$$\frac{a(x, 1, e^2)}{a(x_0, 1, e^2)} \sim (|x-1|)^{-3e^2/2\pi x}(|x_0-1|)^{-3e^2/2\pi x_0};$$

$$\frac{b(x, 1, e^2)}{b(x_0, 1, e^2)} \sim (|x-1|)^{-(3e^2/4\pi)(1+x)/x} \times (|x_0-1|)^{(3e^2/4\pi)(1+x_0)/x_0}.$$

From this we see that the functions a,b, near $k^2 \sim m^2$ have the well known "infrared singularity."

$$a \sim a_0\left(\left|1-\frac{k^2}{m^2}\right|\right)^{-3e^2/2\pi}; \qquad b \sim b_0\left(\left|1-\frac{k^2}{m^2}\right|\right)^{-3e^2/2\pi}. \tag{35}$$

Such a behavior for these functions actually follows directly from (34). Indeed the main part of the integral on the right side of this equation in the region of interest will be:

$$\alpha(e^2)\ln\left|1-\frac{k^2}{m^2}\right|$$

and, for this reason, the main part of the function s will be

$$C\left|1-\frac{k^2}{m^2}\right|^{\alpha(e^2)}; \quad C = \text{const.} \tag{36}$$

We have seen that second order perturbation theory gives for the exponent $\alpha(e^2)$ the value $-3e^2/2\pi$.

We now make some remarks on vertex parts. Let us take for example, the vertex value of Γ with two electron lines and one photon line. The transformation law for Γ in the multiplicative renormalization group, as is well known, will be

$$\Gamma \to \Gamma' = z_2^{-1}\Gamma, \tag{37}$$

whereupon

$$a' = z_2 a. \tag{38}$$

From the other side, under our "λ-normalization", the Γ function, from considerations of single valuedness in momentum space must have the form

$$\Gamma_\lambda = \Gamma\left(\frac{k}{\lambda}, \frac{k-q}{\lambda}, \frac{m}{\lambda}, e_\lambda^2\right).$$

We find, then, from (37) and (38)

$$\Gamma\left(\frac{k}{\lambda_2}, \frac{k-q}{\lambda_2}, \frac{m}{\lambda_2}, e_{\lambda_2}^2\right) \tag{39}$$

$$= \frac{a(1, m^2/\lambda_1^2, e_{\lambda_1}^2)}{a(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_{\lambda_2}^2)} \Gamma\left(\frac{k}{\lambda_1}, \frac{k-q}{\lambda_1}, \frac{m}{\lambda_1}, e_{\lambda_1}^2\right),$$

in which

$$e_{\lambda_1}^2 = e_{\lambda_2}^2 d(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_{\lambda_2}^2).$$

If here we take λ_1^2, to be of the order of the larger of the quantities $|k^2|$, $|(k-q)^2|$, $|q^2|$, we can, using these formulae, study the behavior of Γ for arbitrary values of k, $k-q$, q as the largest of the quantities $|k^2/m^2|$, $|(k-q)^2/m^2|$, $|q^2/m^2|$ gets to be of the order of unity.

We note, in conclusion, that the method of renormalization groups that we have presented can be carried over likewise into meson theories also. For example, let us consider the neutral pseudoscalar theory.

We set:

$$G = \frac{a\hat{p} + bm}{p^2 - m^2}, \quad D = \frac{d}{p^2 - \mu^2}, \quad \Gamma^5(p, p) = \gamma^5\Gamma,$$

where m, μ are the experimental masses of the nucleon and meson. With these designations the renormalization group will be (if we neglect some difficulties connected with the introduction of a direct meson-meson interaction, arising from the presence of the known four vertex part):**

$$a \to a' = z_2 a, \quad b \to b' = z_2 b, \tag{40}$$

$$\Gamma \to \Gamma' = z_2^{-1}\Gamma,$$

$$d \to d' = z_3 d, \quad g^2 \to g'^2 = g^2 (z_1 z_2^{-1})^2 z_3^{-1}.$$

It is convenient, further, to introduce the function

$$\delta = d \cdot a^2 \cdot \Gamma^2, \tag{41}$$

for which

$$g'^2 \delta' = g^2 \delta. \tag{42}$$

We now make use of λ-normalization, and set $d=1$, $\delta=1$ for $p^2 = \lambda^2$. Then, drawing on considerations of single valuedness in momentum space, we note that all functions being studied will depend only on the arguments

$$p^2/\lambda^2, \quad m^2/\lambda^2, \quad \mu^2/\lambda^2, \quad g^2.$$

For this reason, we get from (40) and (41):

$$\delta\left(\frac{k^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, \frac{\mu^2}{\lambda_1^2}, g_1^2\right) = \frac{\delta(k^2/\lambda_2^2, m^2/\lambda_2^2, \mu^2/\lambda_2^2, g_2^2)}{\delta(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, \mu^2/\lambda_2^2, g_2^2)}; \tag{43}$$

$$g_1^2 = g_2^2 \delta(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, \mu^2/\lambda_2^2, g_2^2);$$

$$s\left(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, \frac{\mu^2}{\lambda_2^2}, g_2^2\right) = s\left(1, \frac{m^2}{\lambda_2^2}, \frac{\mu^2}{\lambda_2^2}, g_2^2\right) \frac{s(k^2/\lambda_1^2, m^2/\lambda_1^2, \mu^2/\lambda_1^2, g_1^2)}{s(\lambda_2^2/\lambda_1^2, m^2/\lambda_1^2, \mu^2/\lambda_1^2, g_1^2)},$$

where $s = a, b, \Gamma, d$.

Hence, arguing as before, we get the general Lee equations in the form

** Remark in Proof. An investigation of the renormalization group in meson theory taking into account direct meson-meson interactions has recently been carried out by one of the authors.[6]

[6] D.V.Shirkov, Dokl. Akad. Nauk. SSSR, 105, 972 (1955).

$$\frac{\partial}{\partial x}\left(g^2\delta(x, y, z, g^2)\right) = \frac{g^2\delta(x, y, z, e^2)}{x}\left[\frac{\partial}{\partial \xi}\delta\left(\xi, \frac{y}{x}, \frac{z}{x}, g^2\delta\right)\right]_{\xi=1},$$
$$\frac{\partial}{\partial x}\left[\ln s(x, y, z, g^2)\right] = \frac{1}{x}\left[\frac{\partial}{\partial \xi}\ln s\left(\xi, \frac{y}{x}, \frac{z}{x}, g^2\delta\right)\right]_{\xi=1}. \tag{44}$$

In the region of high momenta, when $|k^2/m^2| \gg 1$, $|k^2/\mu^2| >> 1$, equations (44) can be simplified, just as in the case of electrodynamics, by setting $x = |k^2/m^2|$ and discarding y and z.

We then obtain:

$$\int_{g^2}^{g^2\delta} \frac{dt}{t\varphi(t)} = \ln x; \quad \varphi(t) = \left[\frac{\partial}{\partial \xi}\delta(\xi, t)\right]_{\xi=1};$$

$$\ln\frac{s(x, g^2)}{s(x_0, g^2)} = \int_{e^2 d(x_0, g^2)}^{e^2 d(x, g^2)} \frac{dt}{t\varphi(t)}\left[\frac{\partial}{\partial \xi}\ln s(\xi, t)\right]_{\xi=1},$$

into which we can introduce perturbation theory formulae. Completely analogous equations can be written for charge symmetrical theories.

Translated by A. Turkevich
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