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Dielectric Losses in Ionic Dielectrics in Strong Electric Fields

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Skanavi[1] introduced a representation for dielectrics containing ions which may be found in two neighboring positions of equilibrium in the crystalline lattice, and investigated the losses due to transitions of these ions between the two positions of equilibrium, in weak electric fields. In the present article, dielectric losses due to the same mechanism in strong fields are investigated. Three limiting cases are considered. The results of the calculations are summarized in formulas (33)-(34), (35)-(38), and (44)-(45).

1. GENERAL STATEMENTS

LET the probability of transition of an ion from the first position of equilibrium to the second, or the reverse, per unit time, be $\nu \exp(-U/kT)$[2]. In the presence of an electric field, the energy barrier U changes. Expanding U in a series of powers of the field E, we may write

$$U' = U \mp pE, \tag{1}$$

where the two signs refer to transitions of the ion in opposite directions. If $n_1 = n - \Delta n$, $n_2 = n + \Delta n$ are the number of ions in the two positions of equilibrium, per unit volume, then

$$\frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} = -n_1 \nu \exp\left\{-\frac{U - pE}{kT}\right\} + n_2 \nu \exp\left\{-\frac{U + pE}{kT}\right\}, \tag{2}$$

and the dipole moment per unit volume is

$$P = (n_2 - n_1)p' = 2p'\Delta n, \tag{3}$$

where $p' \approx p$ has the dimensions of a dipole moment, and an order of magnitude of 10^{-18}. By virtue of Eq. (2), P satisfies the equation

$$\frac{\partial P}{\partial t} = \frac{\overline{P} - P}{\tau}; \tag{4}$$

where

$$\overline{P} = np' \tanh \frac{pE}{kT} \equiv P_0 \tanh \frac{pE}{kT}, \tag{5}$$

and

$$\tau^{-1} = 2\nu e^{-U/kT} \cosh \frac{pE}{kT} \equiv \tau_0^{-1} \cosh \frac{pE}{kT}. \tag{6}$$

Introducing the new variable

$$\vartheta = \int_0^t \tau^{-1}(x)\, dx, \tag{7}$$

we obtain the solution of Eq. (4) in the form

$$P(\vartheta) = e^{-\vartheta} \int e^{\vartheta'} \overline{P}(\vartheta')\, d\vartheta' \tag{8}$$

Let

$$E = E_0 \sin \omega t, \tag{9}$$

and

$$pE_0 / kT = \alpha, \quad \omega t = x, \quad \omega\tau_0 = \beta. \tag{10}$$

In order to obtain the losses in the stationary state, we exclude the influence of the instant of application of the field by assuming it to have occurred at $t = -\infty$. Then Eq. (8) gives

$$P(x) = \frac{P_0}{\beta} e^{-\vartheta(x)} \int_{-\infty}^{x} e^{\vartheta(y)} \sinh(\alpha \sin y)\, dy, \tag{11}$$

where

$$\vartheta(x) = \frac{1}{\beta} \int_0^x \cosh(\alpha \sin y)\, dy. \tag{12}$$

Hence, it is evident that the dipole moment depends on the two parameters α and β, Eq. (10) determining the dependence of the losses on the field E and the frequency ω.

As a consequence of the periodicity of the field the energy lost during the period T_0 is

$$q = \int_0^{T_0} E\, dP = \int_0^{2\pi} E \frac{dP}{dx}\, dx = -\int_0^{2\pi} P \frac{dE}{dx}\, dx.$$

Consequently, the losses per unit time are

$$Q = -\frac{\omega}{2\pi} E_0 \int_0^{2\pi} \cos x \cdot P(x)\, dx. \tag{13}$$

[1] G. I. Skanavi, *Physics of Dielectrics*, GITTL, 1949

[2] Ia. I. Frenkel, *Kinetic Theory of Liquids*, Oxford, 1946; p 22.

In the future we will make use of the relation

$$\vartheta(x+n\pi)=\vartheta(x)+\vartheta(n\pi), \tag{14}$$

from which, according to Eq. (11), it follows that

$$P(x+n\pi) = \frac{P_0}{\beta}e^{-\vartheta(x)-\vartheta(n\pi)}\int_{-\infty}^{x+n\pi} e^{\vartheta(y)}\sinh(\alpha\sin y)\,dy = (-1)^n P(x). \tag{15}$$

2. LOSSES AT LOW FREQUENCIES

The methods for calculating $P(x)$ are somewhat different for $\beta \lesssim 1$ and for $\beta \gg 1$, i.e., for low and for high frequencies. In the first case, it is convenient to transform Eq. (11) for $P(x)$ on the basis of Eq. (12) for $\theta(x)$:

$$P=\frac{P_0}{\beta}e^{-\vartheta(x)}\int_{-\infty}^{x} e^{-\vartheta(y)}\{\cosh(\alpha\sin y) - e^{-\alpha\sin y}\}\,dy = P_0\Big[1-\frac{1}{\beta}e^{-\vartheta(x)}\int_{-\infty}^{x} e^{\vartheta(y)-\alpha\sin y}dy\Big] \equiv P_0(1-J), \tag{16}$$

whence, in virtue of Eq. (16), the losses are

$$Q=\frac{\omega}{\pi}E_0P_0\int_0^{\pi}\cos xJ(x)\,dx \equiv Q_1+Q_2, \tag{17}$$

$$Q_1=\frac{\omega}{\pi}E_0P_0\int_0^{\pi/2}\cos xJ(x)\,dx;\qquad Q_2=\frac{\omega}{\pi}E_0P_0\int_{\pi/2}^{\pi}\cos xJ(x)\,dx,$$

where

$$J(x) = \frac{1}{\beta}\int_{-\infty}^{x}\exp[-\{\vartheta(x)-\vartheta(y)+\alpha\sin y\}]\,dy = \frac{1}{\beta}\int_{-\infty}^{0}\exp[-\{\vartheta(x)-\vartheta(x+y) + \alpha\sin(x+y)\}]\,dy = \frac{1}{\beta}\int_0^{\infty}\exp\Big[-\Big\{\alpha\sin(x-y) + \frac{1}{\beta}\int_0^{y}\cosh[\alpha\sin(x-z)]\,dz\Big\}\Big]dy. \tag{18}$$

When $\alpha \gg 1$, the integral in the braces, rapidly increasing with an increase of y, has primary importance. Therefore, for the calculation of J, the essential region is $y \ll 1$, and hence, $z \ll 1$. Expanding $\sin(x-y)$ and $\sin(x-z)$ in a series of powers of y and z, and limiting the series to the first powers, we are able to perform the integration. Considering, on the basis of Eq. (15), the first half of a period ($\sin x > 0$), denoting

$$\alpha\cdot\sin y=\xi,\quad \alpha\beta\cos x=\eta \tag{19}$$

and introducing the variable

$$\tau=\alpha\cdot|\cos x|\cdot y,$$

we obtain:

if $0<\omega t<\pi/2$ or $\eta>0$, (20

$$J=J_1=\frac{1}{\eta}\exp\Big\{-\frac{1}{\eta}\sinh\xi-\xi\Big\}\int_0^{\infty}\exp\Big\{-\frac{1}{\eta}\sinh(\tau-\xi)+\tau\Big\}d\tau,$$

and if $\pi/2<\omega t<\pi$ or $\eta<0$,

$$J=J_2=-\frac{1}{\eta}\exp\Big\{-\frac{1}{\eta}\sinh\xi-\xi\Big\}\int_0^{\infty}\exp\Big\{\frac{1}{\eta}\sinh(\tau+\xi)-\tau\Big\}d\tau.$$

It is easy to verify that when $\xi = 0$ the sum of the two expressions is equal to two, which corresponds [by Eq. (17)] to the condition

$$P(0)+P(\pi)=0, \tag{21}$$

when $x = \pi/2$ both expressions tend toward zero exponentially, so that the region of values of $x \approx \pi/2$ is nonessential, and in the future we exclude it from consideration.

In order to evaluate the conditions of applicability of the expansions in powers of y and z, which led to Eq. (20), it is necessary to perform the expansion in Eq. (18) up to the second order of the small quantities, and to determine the conditions under which the additional terms are insignificant in the essential region. In this way we arrive at the conditions:

for $\alpha = pE_0/kT \gg 1$

$$\text{arc}\sinh^3\alpha\omega\tau_0 \gg \alpha^2, \tag{22}$$

and for $\alpha \lesssim 1$

$$\beta^2 = (\omega\tau_0)^2 \gg 1. \tag{23}$$

3. CALCULATION OF POLARIZATION LOSSES

Let us denote the integrals J_1 and J_2 in Eq. (20) by

$$J = \frac{1}{|\eta|} \exp\left\{\mp \frac{1}{\eta} \sinh\xi - \xi\right\} \times \int_0^\infty \exp\left\{-\frac{1}{|\eta|}\sinh(\tau \mp \xi) \pm \tau\right\} d\tau. \tag{24}$$

They are calculated simply when $|\eta| = \alpha\beta|\cos x| \gg\ll 1$.

Let us consider first the case $|\eta| \gg 1$ or

$$\alpha\beta = \frac{pE_0}{kT}\,\omega\tau_0 \gg 1, \tag{25}$$

which is possible only when the conditions of Eq. (22) are fulfilled.

Let us denote

$$\rho = \operatorname{arc\,sinh} |\eta| \approx \ln 2|\eta|, \quad z = \tau - \rho \mp \xi. \tag{26}$$

Then the expression in the exponent in Eq. (24) becomes

$$-\xi - \frac{1}{|\eta|}\sinh(\tau \mp \xi) \pm \tau = -\frac{\sinh(z+\rho)}{\sinh\rho} \pm (z+\rho) = -\sinh z \coth\rho - \cosh z \pm (z+\rho).$$

But when $|\eta| \gg 1$, $\coth\rho \approx 1$, and we get

$$e^{-z} \pm (z+\rho),$$

Therefore

$$J = \frac{1}{|\eta|} \exp\left\{\mp \frac{1}{|\eta|}\sinh\xi\right\} \times \int_{-\xi-\rho}^\infty \exp\{-(e^z \mp z) \pm \rho\}\, dz.$$

Letting $e^z = t$, we get

$$J_1 = 2\exp\left\{-\frac{1}{|\eta|}\sinh\xi\right\} \int_{e^{-(\xi+\rho)}}^\infty e^{-t}dt = 2\exp\left(-\frac{1}{|\eta|}\left\{\sinh\xi + \frac{1}{2}e^{-\xi}\right\}\right) = 2\exp\left(-\frac{1}{2|\eta|}e^{\xi}\right),$$

$$J_2 = \frac{1}{2\eta^2}\exp\left\{\frac{1}{|\eta|}\sinh\xi\right\} \times \int_{\xi-\rho}^\infty \exp\{-e^z - z\}\, dz. \tag{27}$$

Letting $e^z = t$, we get

$$J_2 = -\frac{1}{2\eta^2}\exp\left\{\frac{1}{|\eta|}\sinh\xi\right\}\int_{e^{\xi-\rho}}^\infty e^{-t}\frac{dt}{t^2} = \frac{1}{|\eta|}\exp\left\{\frac{1}{|\eta|}\sinh\xi - \xi\right\}\exp\left[-(e^{\xi-\rho})\right] + O,$$

where O is a term of the order $\frac{1}{\eta^2}\ln|\eta|$. Disregarding the additional term, we have

$$J_2 = \frac{1}{|\eta|}\exp\left\{-\frac{1}{2|\eta|}e^{-\xi} + \xi\right\}.$$

Finally, disregarding the first of the quantities which are added in the square brackets, we find

$$J_2 = \frac{1}{|\eta|}e^{-\xi}. \tag{28}$$

Formulas (27) and (28) are unvalid only in the neighborhood of $x = \frac{\pi}{2}$, but in this case the integrals J_1 and J_2 tend toward zero exponentially. Finally, we have for $|\eta| \gg 1$:

$$P = P_0\left[1 - 2\exp\left(-\frac{1}{2|\eta|}e^{\xi}\right)\right], \quad \eta > 0, \qquad P = P_0\left[1 - \frac{1}{|\eta|}e^{-\xi}\right], \quad \eta < 0. \tag{29}$$

Let us proceed now to the case $|\eta| \ll 1$, i.e.,

$$\alpha\beta = \frac{pE_0}{kT}\,\omega\tau_0 \ll 1, \tag{30}$$

which corresponds to the low frequencies, $\omega\tau_0 \ll 1$. In this case, in the expression for J, Eq. (24), only small values of τ contribute essentially. Expanding $\sinh(\tau \pm \xi)$ in a series of powers of τ and integrating, we get

$$J = \frac{1}{\cosh\xi} e^{-\xi}\left(1 \pm \frac{|\eta|}{\cosh\xi}\right)$$

$$= \frac{1}{\cosh\xi} e^{-\xi}\left(1 + \frac{\alpha\beta\cos x}{\cosh\xi}\right), \quad (31)$$

$$P(x) = P_0\left\{1 - \left(\frac{1}{\cosh\xi} + \frac{\alpha\beta\cos x}{\cosh^2\xi}\right)e^{-\xi}\right\}. \quad (32)$$

Now let us determine the magnitude of the losses. When $|\eta| \gg 1$ Eq. (29) decreases exponentially with an increase of $\xi = \alpha \sin x$; therefore only small values of x are essential, and we may write $\eta = \alpha\beta\cos x \approx \alpha\beta$. Introducing the variable $\sin x = t$, we get

$$Q = Q_1 + Q_2 = \frac{\omega}{\pi} E_0 P_0$$

$$\times \left\{\int_0^1 2\exp\left(-\frac{1}{2\alpha\beta} e^{\alpha t}\right) dt - \frac{1}{\alpha\beta}\int_0^1 e^{-\alpha t} dt\right\}.$$

After some simple transformations, taking into account $e^{\alpha} \gg 1$, we get

$$Q = \frac{\omega}{\alpha\pi} E_0 P_0 \left\{-2\mathrm{Ei}\left[-\frac{1}{2\alpha\beta}\right] - \frac{1}{\alpha\beta}\right\}.$$

Since, for $x \ll 1$,

$$\mathrm{Ei}[-x] = \ln \gamma x,$$

where $\gamma = e^C = 1.78$ (C is Euler's constant), then

$$Q = \frac{\omega}{\alpha\pi} E_0 P_0 \left\{2\ln\frac{2\alpha\beta}{\gamma} - \frac{1}{\alpha\beta}\right\}.$$

Returning to the original variables, we get (still disregrading the second term)

$$Q = 2\frac{\omega k T n}{\pi}\frac{p'}{p}\ln\left(1.122\frac{pE_0}{kT}\omega\tau_0\right), \quad (33)$$

under the conditions:

$$\frac{pE_0}{kT} \gg 1, \quad \frac{pE_0}{kT}\omega\tau_0 \gg 1,$$

$$\operatorname{arc\,sinh}\frac{pE_0}{kT}\omega\tau_0 \ll \left(\frac{nE_0}{kT}\right)^2. \quad (34)$$

In the opposite limiting case $|\eta| \ll 1$, the first term in Eq. (31) when multiplied by $\cos x$ in accordance with Eq. (17) and integrated from zero to π gives zero. This leaves only the second term:

$$Q = \alpha\beta\,\frac{\omega}{\pi} E_0 P_0 \int_0^{\pi} \frac{\cos^2 x}{\cosh^2\xi} e^{-\xi} dx \quad (35)$$

$$= \frac{2}{\pi} n \frac{p' p E_0^2}{kT}\omega^2\tau_0 f\left(\frac{pE_0}{kT}\right),$$

where

$$f(\alpha) = \int_0^1 \frac{e^{-\alpha x}}{\cosh^2 \alpha x}\sqrt{1 - x^2}\, dx. \quad (36)$$

The function $f(\alpha)$ decreases monotonically with increasing α

$$f(\alpha) = \frac{\pi}{4} - \frac{\alpha}{3} \qquad \text{when } \alpha \ll 1,$$

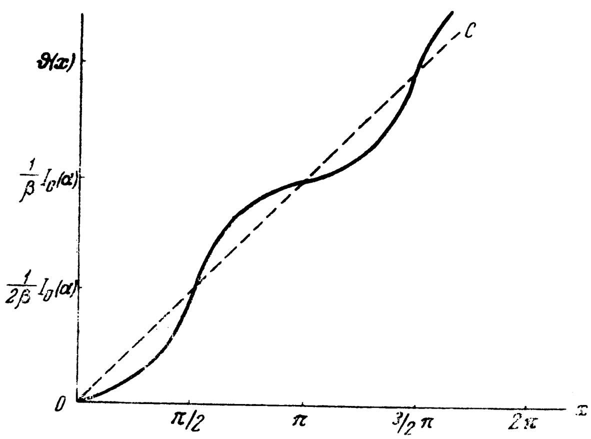
$$f(\alpha) = \frac{1}{\alpha}\left(\frac{\pi}{2} - 1\right) \quad \text{when } \alpha \gg 1. \quad (37)$$

Eqs. (35)-(37) are valid under the conditions:

$$(pE_0/kT)\,\omega\tau_0 \ll 1, \quad \omega\tau_0 \ll 1. \quad (38)$$

4. LOSSES AT HIGH FREQUENCIES

We now return to Eqs. (11) and (12).



The function $\theta(x)$ has the approximate form shown in the Figure, and the slope of thé straight line OC is, according to Eq. (14),

$$\operatorname{tg}\varphi = \frac{1}{\pi}\,\vartheta(\pi) = \frac{1}{\beta} I_0(\alpha),$$

where $I_0(\alpha)$ is a Bessel function with an imaginary argument. Therefore we can set

$$\vartheta(x) = \frac{1}{\beta} I_0(\alpha)\, x + \psi(x), \quad (39)$$

where

$$\psi(x) = \frac{1}{\beta}\int_0^x \{\cosh(\alpha\sin y) \quad (40)$$

$$- I_0(\alpha)\}\, dy \leqslant \frac{1}{\beta} I_0(\alpha)$$

and has a period of π. We set

$$\frac{1}{\pi}\,\vartheta(\pi) = \frac{I_0(\alpha)}{\omega\tau_0} \ll 1, \quad (41)$$

substitute Eq. (39) into Eq. (11) and expand in a series of powers of ψ. Taking into account the inequalities (40) and (41), we have

$$P(x)=\frac{P_0}{\beta}\exp\left\{-\frac{1}{\beta}I_0(\alpha)x\right\}$$

$$\times\left[\int_{-\infty}^{x}\exp\left\{\frac{1}{\beta}I_0(\alpha)y\right\}\sinh(\alpha\sin y)\,dy\right.$$

$$-\psi(x)\int_{\infty}^{x}\exp\left\{\frac{1}{\beta}I_0(\alpha)y\right\}\sinh(\alpha\sin y)\,dy \qquad (42)$$

$$\left.+\int_{\infty}^{x}\exp\left\{\frac{1}{\beta}I_0(\alpha)y\right\}\psi(y)\sinh(\alpha\sin y)\,dy\right]$$

$$\equiv P_1+P_2+P_3.$$

We develop $\psi(xq)$ and sinh (α sin y) in Fourier series:

$$\psi(x)=\sum_{n=1}^{\infty}a_n\sin 2nx,$$

$$\sinh(\alpha\sin y)=\sum_{k=0}^{\infty}b_{2k+1}\sin(2k+1)y,$$

where
$$a_n=\frac{1}{\pi}(-1)^n\frac{1}{\beta}I_{2n}(\alpha),\quad b_{2k+1}=2(-1)^kI_{2k+1}(\alpha),$$

since both functions are odd. Then the losses are equal to

$$Q=-\frac{\omega E_0}{2\pi}\int_{-\pi}^{\pi}P(x)\cos x\,dx \qquad (43)$$

$$=Q_1+Q_2+Q_3,$$

corresponding to the three parts of Eq. (42).

After some simple calculations we get

$$Q_1=E_0P_0\omega\frac{(1/\beta)I_1(\alpha)}{1+(I_0^2(\alpha)/\beta^2)},$$

$$Q_2=\frac{E_0P_0\omega}{\pi^2}\frac{I_0(\alpha)}{\beta^3}\sum_{k=1}^{\infty}I_2\ (\alpha)I_{2k+1}(\alpha)\frac{1}{(I_0^2(\alpha)/\beta^2)+(2k+1)^2},$$

$$Q_3=\frac{E_0P_0\omega}{\pi^2}\frac{I_0(\alpha)}{\beta^3}\sum_{k=1}^{\infty}\frac{I_{2k}(\alpha)}{k}(I_{2k+1}(\alpha)+I_{2k+1}(\alpha))\frac{1}{(I_0^2(\alpha)/\beta^2)+(2k+1)^2}.$$

Since the small parameter in the expansion whose square we are neglecting is $\frac{1}{\beta}I_0(\alpha)$, then to our approximation,

$$Q_1+Q_2+Q_3=E_0P_0\omega\cdot\frac{1}{\beta}I_1(\alpha),$$

and consequently

$$Q=\frac{E_0P_0}{\tau_0}I_1\left(\frac{pE_0}{kT}\right), \qquad (44)$$

under the condition

$$\frac{I_0^2(pE_0/kT)}{\omega^2\tau_0^2}\ll 1. \qquad (45)$$

From Eqs. (35) and (44) we see that, just as in the case of a weak field[1], the losses increase with frequency at low frequencies and are independent of frequency at high frequencies.

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