Tensors Which are Characterized by Two Real Spinors

G. A. ZAITSEV

(Submitted to JETP editor April 3, 1954)
J. Exper. Theoret. Phys. USSR 29, 166-175 (August, 1955)

Formulas are obtained which enable one simply to determine a real spinor in terms of the primary tensor characterizing it. With the aid of these formulas we establish the relation between two spinors, corresponding to two given triples E, H, j. Tensors whose components are expressed in terms of two real spinors are investigated. Two types of spinor transformations are introduced, corresponding to different possible interpretations of the gauge transformation. The significance of the spinor transformations is established; it is shown that if we regard the group of tensors as initially defined, then we can with their aid find two real spinors to within a spinor transformation of one of the types studied. The components of the initial tensors can be expressed in terms of the spinors, in which case they will not change when the corresponding spinor transformation is carried out.

IN references 1 and 2 we studied some of the properties of real spinors --- systems of parameters defined by a four-dimensional antisymmetric tensor for which both invariants are equal to zero. In references 1, 3 and 4, the question of the use of real spinors in certain physical problems was considered. For further development of the ideas introduced in these papers it is necessary to investigate the more general case of tensors characterized by a pair of real spinors. The solution of this problem is the purpose of the present work. The notation used in the paper corresponds to that of the articles mentioned; we shall also make continual use of the properties of matrix-tensors considered in reference 4.

1. FUNDAMENTAL FORMULAS

A single real spinor ψ determines the matrix-tensors

$$F=\frac{1}{2}R_\alpha R_\beta F^{\alpha\beta}=R\underline{\mathbf{H}}+R_4\underline{\mathbf{E}},\qquad(1)$$

$$P=R_\alpha \vec{j^\alpha}=\underline{\mathbf{j}}+\rho R_4;$$

where (cf references 2, 3 and 5):

$$\mathbf{E}=E_iR^i,\quad \mathbf{H}=H_iR^i,\qquad(2)$$

$$\rho^2=\mathbf{E}^2=\mathbf{H}^2,\quad \mathbf{j}=-(1/\rho)R\mathbf{E}\,\mathbf{H};$$

$$E_i=-\psi'JR^i\psi,\quad H_i=\psi'R^i\psi,\qquad(3)$$

$$j^i=\psi'R_4R^i\psi,\quad \rho=\psi'\psi.$$

Since E and H are primary quantities, while ψ is a derived quantity, it is necessary to find a sufficiently simple method enabling one to find the components of the associated real spinor ψ if the E and H are given.

Suppose that for some matrix, such that $\mathbf{d}=d_iR^i$, $\mathbf{d}^2=\rho^2$, the relation

$$\mathbf{d}\psi=\rho\psi.\qquad(4)$$

is satisfied.

Multiplying Eq. (4) on the left by ψ', $-\psi'J$, $\psi'R_4$, and using Eq. (3), we get

$$(\underline{\mathbf{H}}\,\underline{\mathbf{d}})=\rho^2,(\underline{\mathbf{E}}\,\underline{\mathbf{d}})=(\underline{\mathbf{j}}\,\underline{\mathbf{d}})=0,\qquad(5)$$

so that, since E, H and j are mutually orthogonal and have equal length, we have d = H. We thus find:

$$\mathbf{H}\psi=\rho\psi.\qquad(6)$$

It is not difficult to see that if the components H_i are related to ψ by formulas of the type of Eq. (3), then formula (6) will be valid.

In precisely the same way we get

$$-J\mathbf{E}\psi=\rho\psi.\qquad(7)$$

Using the fact that EH $=\rho R\,\mathbf{j}$ and multiplying both sides of (6) from the left by $-J\mathbf{E}$, we have as a consequence

[1] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 653 (1953)

[2] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 667 (1953)

[3] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 675 (1953)

[4] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 524 (1955); Soviet Phys. 1, 411 (1955)

[5] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 530 (1955); Soviet Phys. 1, 491 (1955)

$$-R^4\mathbf{j}\,\psi = R_4\mathbf{j}\,\psi = \rho\psi. \tag{8}$$

If E and H are given, the corresponding real spinor ψ can be found from Eqs. (6) and (7). (It is understood that the three-dimensional "vectors" E and H must have equal length and be orthogonal. Otherwise, these equations will be incompatible.)

Formulas (6), (7) and (8), which we shall call the fundamental formulas for a single real spinor, will be very important for what follows.

We observe that it follows from Eq. (8) that

$$P\psi = 0, \tag{9}$$

and from Eqs. (6) and (7) that $F\psi = 0$, where the last relation is obtained from Eq. (9) if we multiply both sides from the left by the matrix $-(1/\rho)\,\mathrm{E}$. Also, if the relation $D\psi = 0$ holds for any matrix four-vector D, then the matrix D is proportional to P.

On the basis of formulas (6)-(8), we consider the question of the relation of two real spinors defined by two different antisymmetric tensors of the second rank having invariants equal to zero. In doing this we shall assume that the position of one triple, E, H, j, relative to the other triple characterizing the second tensor, is known; i.e., we assume, for example, that we know the Euler angles characterizing the rotation of the one triple relative to the other, and also the ratio $\rho_{(2)}/\rho_{(1)}$ of the lengths of the vectors. When we deal with several real spinors, we shall distinguish them by an additional index in parentheses, e.g., $\psi_{(1)}$, $\psi_{(2)}$, etc. We shall also give the index in parentheses for the corresponding vectors E, H, j. In our case we can use the designations 1 and 2.

The real spinor $\psi_{(2)}$ can be obtained from $\psi_{(1)}$ by rotation or reflection of the four-dimensional space and subsequent multiplication by some real number. We shall make use of the fact that the matrix of an arbitrary rotation or reflection can be written in the form of a product of matrices which are linear combinations of the R^α (cf reference 2). Since each of the three spatial basis vectors can be represented as a linear combination of the three linearly independent spatial vectors E, H, j, (strictly speaking, E and H do not behave like the spatial parts of vectors under a general Lorentz transformation, but this does not matter for the present case), the matrices R^i can be expressed as linear combinations of the matrices E, H, j, corresponding to an arbitrary real spinor ψ. Using formulas (6)-(8) we find that under a four-dimensional symmetry transformation, and consequently, also under an arbitrary rotation and reflection, a real spinor ψ will be multiplied by matrices of the type

$$K = k_4 + k_1R_4 + k_2R + k_3J. \tag{10}$$

In this way we can establish that $\psi_{(2)}$ is related to $\psi_{(1)}$ by the formula

$$\psi_{(2)} = K\psi_{(1)}, \tag{11}$$

where the matrix K can be written in the form of Eq. (10).

Let us study in more detail the question of the form of the parameters k_α. First of all, we note that matrices of the type of Eq. (10) are isomorphic to quaternions; to the matrices R_4, R, J, there correspond the quaternion units i, j, k. In particular,

$$K'K = k_1^2 + k_2^2 + k_3^2 + k_4^2 = \tau^2 \tag{12}$$

can be regarded as the norm of the quaternion, so that it follows from Eq. (11) that

$$\rho_{(2)} = \tau^2\rho_{(1)}. \tag{13}$$

Using formulas (3), (10) and (11), we express $\mathrm{E}_{(2)}$, $\mathrm{H}_{(2)}$ and $\mathrm{j}_{(2)}$ in terms of $\mathrm{E}_{(1)}$, $\mathrm{H}_{(1)}$, $\mathrm{j}_{(1)}$, and the parameters k_α. After simple calculations we obtain

$$\mathbf{E}_{(2)} = (k_4^2 + k_1^2 - k_2^2 - k_3^2)\,\mathbf{E}_{(1)} + (-2k_4k_3 + 2k_1k_2)\,\mathbf{H}_{(1)} + (2k_4k_2 + 2k_1k_3)\,\mathbf{j}_{(1)}, \tag{14}$$

$$\mathbf{H}_{(2)} = (2k_4k_3 + 2k_1k_2)\,\mathbf{E}_{(1)} + (k_4^2 - k_1^2 + k_2^2 - k_3^2)\,\mathbf{H}_{(1)} + (-2k_4k_1 + 2k_2k_3)\,\mathbf{j}_{(1)}, \tag{15}$$

$$\mathbf{j}_{(2)} = (-2k_4k_2 + 2k_1k_3)\,\mathbf{E}_{(1)} + (2k_4k_1 + 2k_2k_3)\,\mathbf{H}_{(1)} + (k_4^2 - k_1^2 - k_2^2 + k_3^2)\,\mathbf{j}_{(1)}. \tag{16}$$

From Eqs. (14)-(16) it is evident that the k_α coincide with the well-known Rodrigues parameters, relating one triple of mutually orthogonal vectors of equal length with a second triple (cf, for example, reference 6, pp. 183-184, where k_4, k_1, k_2

[6] W. D. MacMillan, *Dynamics of Rigid Bodies*, McGraw-Hill, 1936

and k_3 are denoted by the symbols ρ, λ, μ and ν)*. From them we obtain the connection of the parameters k_α to the Euler angles characterizing the rotation of the triple $\mathbf{E}_{(2)}$, $\mathbf{H}_{(2)}$, $\mathbf{j}_{(2)}$ relative to $\mathbf{E}_{(1)}$, $\mathbf{H}_{(1)}$, $\mathbf{j}_{(1)}$

$$k_4 = -\tau \cos\frac{1}{2}\vartheta \cos\frac{1}{2}(\psi+\varphi), \tag{17}$$
$$k_1 = \tau \sin\frac{1}{2}\vartheta \cos\frac{1}{2}(\psi-\varphi),$$
$$k_2 = \tau \sin\frac{1}{2}\vartheta \sin\frac{1}{2}(\psi-\varphi),$$
$$k_3 = \tau \cos\frac{1}{2}\vartheta \sin\frac{1}{2}(\psi+\varphi).$$

The Euler angles are chosen so that the angle between the vectors $\mathbf{j}_{(2)}$, $\mathbf{j}_{(1)}$ corresponds to the angle between the z axes of the coordinate systems considered in the kinematics of rigid bodies, the angle between $\mathbf{E}_{(2)}$ and $\mathbf{E}_{(1)}$ corresponds to the angle between the x axes, and the angle between $\mathbf{H}_{(2)}$ and $\mathbf{H}_{(1)}$ to the angle between the y axes.

Formulas (17) enable us to understand the meaning of the parameters k_α. Here we should note that, for given $P_{(1)}$, $F_{(1)}$, and $P_{(2)}$, $F_{(2)}$, the matrix K is determined only to within a sign, as follows from the properties of real spinors (which are determined only to within a common sign). However, another approach to this problem is possible, if we start from other primary tensors, to which we shall return later.

2. TENSORS EXPRESSED IN TERMS OF TWO REAL SPINORS

Let us turn to the important question of the various tensors whose components are expressible in terms of the components of two real spinors. Here we shall use the representation of a tensor in matrix form.

First of all, we can form various matrix four-vectors and matrix antisymmetric tensors of the second rank by taking linear combinations of the matrices $P_{(1)}$, $P_{(2)}$ and $F_{(1)}$, $F_{(2)}$. In particular, we set

$$P_{(+)} = P_{(1)} + P_{(2)}, \tag{18}$$

$$F_{(+)} = F_{(1)} + F_{(2)}. \tag{19}$$

In addition, we can obtain, from two real spinors, tensors formed in another way. From the law of transformation of real spinors, considered in reference 2, it follows that

$$\Omega_1 = \psi'_{(2)} R \psi_{(1)} \tag{20}$$

is invariant, not changing its form under any rotations or reflections of four-dimensional space. (For example, under the symmetry transformation A_1, $A_1^2 = \pm 1$, Ω_1 goes over into $\pm \psi'_{(2)} A_1' R A_1 \psi = \Omega_1$, etc.) We observe that the ambiguity of sign of the matrix K can be eliminated by requiring, say, that the invariant Ω_1 have a definite sign.

We also introduce the matrix-pseudoscalar

$$S = \Omega_2 J, \qquad \Omega_2 = \psi'_{(2)} R_4 \psi_{(1)}, \tag{21}$$

whereupon it is easy to see that Ω_2 is a pseudo-invariant, changing sign under reflections of the four-dimensional space and remaining unchanged under four-dimensional rotations. Using formulas (10), (17) and (13), we can write the following expressions for Ω_1 and Ω_2:

$$\Omega_1 = k_2\rho_{(1)} = \tau\rho_{(1)} \sin\frac{1}{2}\vartheta \sin\frac{1}{2}(\psi-\varphi) \tag{22}$$
$$= \sqrt{\rho_{(1)}\rho_{(2)}} \sin\frac{1}{2}\vartheta \sin\frac{1}{2}(\psi-\varphi),$$

$$\Omega_2 = k_1\rho_{(1)} = \tau\rho_{(1)} \sin\frac{1}{2}\vartheta \cos\frac{1}{2}(\psi-\varphi) \tag{23}$$
$$= \sqrt{\rho_{(1)}\rho_{(2)}} \sin\frac{1}{2}\vartheta \cos\frac{1}{2}(\psi-\varphi).$$

From the law of transformation of real spinors, it further follows that we can also introduce a four-dimensional vector, a pseudovector and an antisymmetric tensor, defined according to the formulas:

$$q^\alpha = \psi'_{(2)} R_4 R^\alpha \psi_{(1)}, \qquad Q = q^\alpha R_\alpha = \mathbf{q} + q^4 R_4, \tag{24}$$

$$n^\alpha = \psi'_{(2)} R R^\alpha \psi_{(1)}, \qquad N = J n^\alpha R_\alpha = J\mathbf{n} + n^4 R, \tag{25}$$

$$\Phi^{\alpha\beta} = -\psi'_{(2)} R^\alpha R^\beta \psi_{(1)},$$
$$\Phi = R\,\mathbf{B} + R_4 \mathbf{D}, \quad B_i = \psi'_{(2)} R^i \psi_{(1)},$$
$$D_i = -\psi'_{(2)} J R^i \psi_{(1)}. \tag{26}$$

Using (10), we can express all these quantities in terms of k_α, $\rho_{(1)}$ and the components of $\mathbf{E}_{(1)}$, $\mathbf{H}_{(1)}$ and $\mathbf{j}_{(1)}$. Thus, for example,

$$\mathbf{q} = -k_2 \mathbf{E}_{(1)} + k_1 \mathbf{H}_{(1)} + k_4 \mathbf{j}_{(1)}, \quad q^4 = k_4 \rho_{(1)}, \tag{27}$$

$$\mathbf{n} = k_1 \mathbf{E}_{(1)} + k_2 \mathbf{H}_{(1)} + k_3 \mathbf{j}_{(1)}, \quad n^4 = k_3 \rho_{(1)}, \tag{28}$$

$$\mathbf{B} = k_3 \mathbf{E}_{(1)} + k_4 \mathbf{H}_{(1)} - k_1 \mathbf{j}_{(1)}, \tag{29}$$

* In reference 6 there is a typographical error in the equation for $\tau^2 \beta_3$; there should be a plus sign on the right instead of a minus sign.

$$\mathbf{D} = k_4 \mathbf{E}_{(1)} - k_3 \mathbf{H}_{(1)} + k_2 \mathbf{j}_{(1)}.$$

This in turn enables us easily to find the relations between different tensors, to which we shall return later.

If we make use of the operation of differentiation, then we can obtain, from two real spinors, another whole set of four-dimensional tensors. For example, the quantities $\frac{\partial \psi'_{(1)}}{\partial x^\alpha} R \psi_{(1)}$ are components of a four-dimensional vector, etc.

We see that we can, from two real spinors $\psi_{(1)}$ and $\psi_{(2)}$, obtain a large number of different tensors. In establishing relations between them and in studying their properties, it is expedient to look at groups of tensors which are related through some common property. As such properties we consider two types of transformations of real spinors, and shall distinguish the set of tensors whose components remain invariant with respect to one of these transformations.

A spinor transformation of the first type is one for which simultaneously $\psi_{(1)}$ changes to $[\exp(Jf)]\psi_{(1)}$ and $\psi_{(2)}$ to $[\exp(-Jf)]\psi_{(2)}$, where f is an arbitrary pseudo-invariant function. It corresponds to a gauge transformation in the form introduced in reference 5 in considering systems of relativistically invariant differential equations of first order for two real spinors. It is not difficult to see that under such a transformation, Ω_1, Ω_2 and the components of $P_{(1)}$, $P_{(2)}$ and Φ remain unchanged. There are no other tensors which are expressed linearly in terms of two real spinors (and do not contain their derivatives with respect to x^α).

We say that a spinor transformation is of the second type if it changes $\psi_{(1)}$ to $\cos f\psi_{(1)} + \sin f\psi_{(2)}$ and $\psi_{(2)}$ to $\cos f\psi_{(2)} - \sin f\psi_{(1)}$, where f is an arbitrary invariant function. Under transformations of the second type, Ω_1, Ω_2 and the components of the matrix-vector $P_{(+)}$, the matrix-pseudovector N and the matrix-tensor of second rank $F_{(+)}$ do not change. The transformation we are considering will be shown in a later paper to correspond to the gauge transformation of the wave function which appears in the Dirac equation for the electron. In studying quantities which are invariant with respect to this transformation, it is convenient to use the notation:

$$\psi_{(+)} = \psi_{(1)} + i\psi_{(2)}, \quad \psi^*_{(+)} = \psi'_{(1)} - i\psi'_{(2)}. \tag{30}$$

Then

$$\Omega_1 = \frac{1}{2} i\psi^*_{(+)} R\psi_{(+)}, \quad \Omega_2 = \frac{1}{2} i\psi^*_{(+)} R_4\psi_{(+)}, \tag{31}$$

$$F^{\alpha\beta}_{(+)} = -\psi^*_{(+)} RR^\alpha R^\beta \psi_{(+)}, \tag{32}$$

$$P^\alpha_{(+)} = j^\alpha_{(1)} + j^\alpha_{(2)} = \psi^*_{(+)} R_4 R^\alpha \psi_{(+)}, \tag{33}$$

$$n^\alpha = \frac{1}{2} i\psi^*_{(+)} RR^\alpha \psi_{(+)}. \tag{34}$$

Of course, the imaginary unit i is introduced only for convenience of notation, and need not be used.

3. PROPERTIES OF TENSORS WHICH ARE INVARIANT WITH RESPECT TO SPINOR TRANSFORMATIONS OF THE FIRST TYPE

We first consider the question of the properties of the quantities Ω_1, $S = \Omega_2 J$, $P_{(1)}$, $P_{(2)}$, Φ, which do not change under spinor transformations of the first type.

From Eq. (29) we have

$$\mathbf{B}^2 - \mathbf{D}^2 = \Omega_2^2 - \Omega_1^2, \quad (\mathbf{BD}) = -\Omega_1\Omega_2, \tag{35}$$

i.e.,

$$\Phi^2 = \Omega_1^2 - \Omega_2^2 - 2\Omega_1\Omega_2 J \tag{36}$$

$$= (\Omega_1 - \Omega_2 J)^2 = (\Omega_1 - S)^2.$$

As for the matrices $P_{(1)}$ and $P_{(2)}$, their squares are obviously equal to zero,

$$P^2_{(1)} = P^2_{(2)} = 0. \tag{37}$$

We calculate the products of these matrix-vectors. According to reference 4,

$$P_{(1)}P_{(2)} = (\mathbf{j}_{(1)}\mathbf{j}_{(2)}) - \rho_{(1)}\rho_{(2)} \tag{38}$$

$$+ R[\mathbf{j}_{(1)}\mathbf{j}_{(2)}] + R_4(\rho_{(1)}\mathbf{j}_{(2)} - \rho_{(2)}\mathbf{j}_{(1)}).$$

Furthermore, in accordance with Eqs. (13), (16) and (29),

$$(\mathbf{j}_{(1)}\mathbf{j}_{(2)}) - \rho_{(1)}\rho_{(2)} = -2(\Omega_1^2 + \Omega_2^2), \tag{39}$$

$$[\mathbf{j}_{(1)}\mathbf{j}_{(2)}] = -2\Omega_1 \mathbf{B} - 2\Omega_2 \mathbf{D}, \tag{40}$$

$$\rho_{(1)}\mathbf{j}_{(2)} - \rho_{(2)}\mathbf{j}_{(1)} = -2\Omega_1 \mathbf{D} + 2\Omega_2 \mathbf{B}, \tag{41}$$

so that

$$P_{(1)}P_{(2)} = -2(\Omega_1^2 + \Omega_2^2) - 2(\Omega_1 + \Omega_2 J)\Phi, \tag{42}$$

$$P_{(2)}P_{(1)} = -2(\Omega_1^2 + \Omega_2^2) + 2(\Omega_1 + \Omega_2 J)\Phi. \tag{43}$$

In addition, we get from Eqs. (26) and (16)

$$[\mathbf{BD}] = -\frac{1}{2}(\rho_{(1)}\mathbf{j}_{(2)} + \rho_{(2)}\mathbf{j}_{(1)}), \tag{44}$$

so that

$$\rho_{(1)}\mathbf{j}_{(2)} = -\Omega_1\mathbf{D} + \Omega_2\mathbf{B} - [\mathbf{BD}], \tag{45}$$

$$\rho_{(2)}\mathbf{j}_{(1)} = \Omega_1\mathbf{D} - \Omega_2\mathbf{B} - [\mathbf{BD}].$$

Thus, assignment of Φ, Ω_1 and Ω_2 determines the directions of the vectors $\mathbf{j}_{(1)}$ and $\mathbf{j}_{(2)}$, but not their lengths $\rho_{(1)}$ and $\rho_{(2)}$ (since the components of Φ do not change under simultaneous multiplication of $\psi_{(1)}$ by some number and division of $\psi_{(2)}$ by that same number, etc.). On the other hand, according to Eqs. (42) and (43), the matrix-tensor Φ is uniquely determined by giving $P_{(1)}$, $P_{(2)}$, Ω_1 and Ω_2, so long as Ω_1 and Ω_2 are not simultaneously equal to zero.

Multiplying both sides of Eqs. (42) and (43) on the left and on the right by the matrices $P_{(1)}$ and $P_{(2)}$, we obtain

$$\Phi P_{(1)} = (\Omega_1 - \Omega_2 J) P_{(1)}, \tag{46}$$

$$\Phi P_{(2)} = -(\Omega_1 - \Omega_2 J) P_{(2)},$$

$$P_{(1)}\Phi = -P_{(1)}(\Omega_1 - \Omega_2 J),$$

$$P_{(2)}\Phi = P_{(2)}(\Omega_1 - \Omega_2 J).$$

Now let us settle the question of the effect on $\psi_{(1)}$ and $\psi_{(2)}$ of various matrix-tensors which are invariant with respect to spinor transformations of the first type. From Eqs. (26) and (29) it follows that

$$\Phi\psi_{(1)} = (R\mathbf{B} + R_4\mathbf{D})\psi_{(1)} = (\Omega_1 - \Omega_2 J)\psi_{(1)}. \tag{47}$$

Similarly, we get

$$\Phi\psi_{(2)} = -(\Omega_1 - \Omega_2 J)\psi_{(2)}, \tag{48}$$

whose validity can be established by interchanging $\psi_{(1)\alpha}$ and $\psi_{(2)\alpha}$ in Eq. (47).

Furthermore, we find from Eqs. (10) and (16)

$$P_{(2)}\psi_{(1)} = 2(\Omega_2 - \Omega_1 J)\psi_{(2)} \tag{49}$$

$$= -2J(\Omega_1 + \Omega_2 J)\psi_{(2)}.$$

Finally, interchanging $\psi_{(1)\alpha}$ and $\psi_{(2)\alpha}$ in Eq. (49), we will have

$$P_{(1)}\psi_{(2)} = -2(\Omega_2 - \Omega_1 J)\psi_{(1)} \tag{50}$$

$$= 2J(\Omega_1 + \Omega_2 J)\psi_{(1)}.$$

It is easy to see that the relations (47)-(50) are not changed by a spinor transformation of the first type.

Up to now we have started from the two spinors $\psi_{(1)}$ and $\psi_{(2)}$ and have found, with their aid, tensors which are invariant under spinor transformations of the first type. A completely different approach is possible here. We could consider that the primary quantities are $P_{(1)}$, $P_{(2)}$, Ω_1 and Φ, connected by the relations given above [cf Eqs. (35)-(46)]*. We define the real spinors $\psi_{(1)}$ and $\psi_{(2)}$ as columns of four numbers which are found from Eqs. (49) and (50). Then, after multiplying both sides of Eqs. (49) and (50) on the left by $P_{(1)}$ and $P_{(2)}$, we get Eqs. (47) and (48).

We show that if we determine $\psi_{(1)}$ and $\psi_{(2)}$ in accordance with Eqs. (49) and (50), then by a suitable choice of the common factor in $\psi_{(1)}$ and $\psi_{(2)}$, we shall have

$$j^{\alpha}_{(1)} = P^{\alpha}_{(1)} = \psi'_{(1)}R_4R^{\alpha}\psi_{(1)}, \tag{51}$$

$$j^{\alpha}_{(2)} = \psi'_{(2)}R_4R^{\alpha}\psi_{(2)}, \quad \Omega_1 = \psi'_{(2)}R\psi_{(1)},$$

$$\Omega_2 = \psi'_{(2)}R_4\psi_{(1)}, \quad \Phi^{\alpha\beta} = -\psi'_{(2)}RR^{\alpha}R^{\beta}\psi_{(1)}.$$

For the proof of this important theorem, we find any solution of Eqs. (49) and (50). From these $\psi_{(1)}$ and $\psi_{(2)}$, we can, by using formulas like (51), find associated quantities which are unchanged by transformations of the first type, and which we designate as $\tilde{j}^{\alpha}_{(1)}$, $\tilde{\Omega}_1$, etc. [Formulas (35)-(46) also enable us to write similar relations for $\tilde{P}_{(1)}$, $\tilde{P}_{(2)}$, $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ and $\tilde{\Phi}$] Multiplying Eq. (50) on the left by $\psi'_{(2)}R$ we get $\Omega_1/\tilde{\Omega}_1 = \Omega_2/\tilde{\Omega}_2 = C$. We shall choose the common factor in $\psi_{(1)}$ and $\psi_{(2)}$, so that $C = 1$. Then from Eqs. (49) and (50) and the corresponding equations obtained by replacing $P_{(1)}$ and $P_{(2)}$ by $\tilde{P}_{(1)}$ and $\tilde{P}_{(2)}$, we have

$$(P_{(2)} - \tilde{P}_{(2)})\psi_{(1)} = 0, \quad (P_{(1)} - \tilde{P}_{(1)})\psi_{(2)} = 0.$$

From the first equation it follows that $P_{(2)} - \tilde{P}_{(2)} = C'\tilde{P}_{(1)}$. Since $\tilde{P}^2_{(1)} = 0$, $P^2_{(2)} = 0$, $C' = 0$, and $P_{(2)} = \tilde{P}_{(2)}$. Similarly, we verify that $P_{(1)} = \tilde{P}_{(1)}$, from which we finally get $\Phi = \tilde{\Phi}$, since Φ is expressed in terms of $P_{(1)}$, $P_{(2)}$, Ω_1 and Ω_2. Thus our assertion is completely demonstrated.

---

* In this case, as pointed out above, Φ is determined by the assignment of other quantities.

As for the law of transformation of $\psi_{(1)}$ and $\psi_{(2)}$, which are defined according to Eqs. (49) and (50), or equivalently from Eq. (51), in order to preserve the tensor character of the initial quantities, it is necessary to suppose that $\psi_{(1)}$ and $\psi_{(2)}$ transform like real spinors, defined according to reference 2, but only to within a transformation of the first type (which is unimportant, since it does not change the fundamental initial quantities). We arrive at the conclusion that if in any problem we have to deal with a pair of real spinors, and if all the fundamental formulas and physically meaningful quantities are unchanged by spinor transformations of the first type, then we may regard the quantities $P_{(1)}, P_{(2)}, \Omega_1, \Omega_2, \Phi$ as the initial quantities in the problem, and the real spinors will be parameters determined by assigning the initial quantities.

In reference 5 we saw that under a spinor transformation of the first type, the components of the vector potential are changed to $A_\alpha - (\hbar c/e) \times \partial f/\partial x^\alpha$. If we make use of the components A_α, which change under a spinor transformation of the first type in this fashion, then we can construct another sequence of tensors containing products of x^α and A_α, whose components are not changed by such transformations. It is easy to verify that the following quantities are unchanged by such a transformation:

$$e_\alpha = \psi'_{(1)} R \frac{\partial \psi_{(2)}}{\partial x^\alpha} + \frac{e}{\hbar c} \Omega_2 A_\alpha, \tag{52}$$

$$m_\alpha = \psi'_{(1)} R_4 \frac{\partial \psi_{(2)}}{\partial x^\alpha} - \frac{e}{\hbar c} \Omega_1 A_\alpha, \tag{53}$$

$$T^{(1)}_{\alpha\beta} = \psi'_{(1)} R R_\alpha \frac{\partial \psi_{(1)}}{\partial x^\beta} + \frac{e}{\hbar c} \psi'_{(1)} R^4 R_\alpha \psi_{(1)} A_\beta, \tag{54}$$

$$T^{(2)}_{\alpha\beta} = \psi'_{(2)} R R_\alpha \frac{\partial \psi_{(2)}}{\partial x^\alpha} - \frac{e}{\hbar c} \psi'_{(2)} R^4 R_\alpha \psi_{(2)} A_\beta. \tag{55}$$

Here e_α and m_α are the components of a vector and a pseudovector, $T^{(1)}_{\alpha\beta}$ and $T^{(2)}_{\alpha\beta}$ are the components of pseudotensors of the second rank. They are connected to the other quantities considered in this section and to one another by certain relations. Thus, from Eqs. (52) and (53) we get

$$\Omega_1 e_\alpha + \Omega_2 m_\alpha = \Omega_1 \psi'_{(1)} R \frac{\partial \psi_{(2)}}{\partial x^\alpha} \tag{56}$$

$$+ \Omega_2 \psi'_{(1)} R_4 \frac{\partial \psi_{(2)}}{\partial x^\alpha} = -\frac{1}{2} \frac{\partial (\Omega_1^2 + \Omega_2^2)}{\partial x^\alpha}$$

$$- \Omega_1 \frac{\partial \psi'_{(1)}}{\partial x^\alpha} R \psi_{(2)} - \Omega_2 \frac{\partial \psi'_{(1)}}{\partial x^\alpha} R_4 \psi_{(2)}$$

$$= -\frac{1}{4} \frac{\partial (\Omega_1^2 + \Omega_2^2)}{\partial x^\alpha} + \frac{1}{4} \frac{\partial \psi'_{(2)}}{\partial x^\alpha} RJP_{(1)} \psi_{(2)}$$

$$-\frac{1}{4} \frac{\partial \psi'_{(1)}}{\partial x^\alpha} RJP_{(2)} \psi_{(1)} = -\frac{1}{4} \frac{\partial (\Omega_1^2 + \Omega_2^2)}{\partial x^\alpha}.$$

Furthermore, we have from Eqs. (54) and (55)

$$j^\alpha_{(1)} T^{(1)}_{\alpha\beta} = 0, \quad j^\alpha_{(2)} T^{(2)}_{\alpha\beta} = 0, \tag{57}$$

from which, taking account of Eqs. (49) and (50) we find

$$(j^\alpha_{(1)} + j^\alpha_{(2)})(T^{(1)}_{\alpha\beta} + T^{(2)}_{\alpha\beta}) = 2\Omega_2 \frac{\partial \Omega_1}{\partial x^\beta} - 2\Omega_1 \frac{\partial \Omega_2}{\partial x^\beta}. \tag{58}$$

4. PROPERTIES OF TENSORS WHICH ARE INVARIANT WITH RESPECT TO SPINOR TRANSFORMATIONS OF THE SECOND TYPE

Just as in the previous section, we can also consider the properties of quantities which are invariant with respect to spinor transformations of the second type. From Eqs. (14) and (15) we obtain

$$(F_{(1)} + F_{(2)})^2 = F^2_{(+)} \tag{59}$$

$$= 4(\Omega_2 + \Omega_1 J)^2 = -4(\Omega_1 - \Omega_2 J)^2.$$

In addition, we obtain from Eq. (28) and Eqs. (14)-(16) by simple calculations,

$$-4N^2 = P^2_{(+)} = -4(\Omega_1^2 + \Omega_2^2), \tag{60}$$

$$\mathbf{n}(\mathbf{j}_{(1)} + \mathbf{j}_{(2)}) - n^4 \rho_{(+)} = 0, \tag{61}$$

$$[\mathbf{n}, \mathbf{j}_{(1)} + \mathbf{j}_{(2)}] = [\mathbf{n}, \mathbf{j}_{(+)}] = \Omega_1 \mathbf{E}_{(+)} - \Omega_2 \mathbf{H}_{(+)}, \tag{62}$$

$$n^4 \mathbf{j}_{(+)} - \rho_{(+)} \mathbf{n} = -\Omega_2 \mathbf{E}_{(+)} - \Omega_1 \mathbf{H}_{(+)}, \tag{63}$$

$$[\mathbf{E}_{(+)}, \mathbf{H}_{(+)}] = \rho_{(+)} \mathbf{j}_{(+)} - 4n^4 \mathbf{n}. \tag{64}$$

Then, since

$$-JNP_{(+)} = (\mathbf{n}\mathbf{j}_{(+)}) - n^4 \rho_{(+)} + R[\mathbf{n}\mathbf{j}_{(+)}] + R_4(n^4 \mathbf{j}_{(+)} - \rho_{(+)} \mathbf{n}),$$

we shall have

$$NP_{(+)} = P_{(+)} N = -(\Omega_1 + \Omega_2 J) F_{(+)}. \tag{65}$$

Using Eq. (60) we get, from Eq. (65),

$$P_{(+)} F_{(+)} = 4(\Omega_1 + \Omega_2 J) N, \tag{66}$$

$$NF_{(+)} = -(\Omega_1 + \Omega_2 J) P_{(+)}.$$

From Eq. (65) it follows that if we know Ω_1, Ω_2,

$P_{(+)}$ and N, we can find the matrix-tensor $F_{(+)}$. On the other hand, the matrices $P_{(+)}$ and N are not uniquely determined by the assignment of $F_{(+)}$, Ω_1 and Ω_2. This follows from the fact that the relations (60), (61) and (65) remain the same if we replace $P_{(+)}$ and $2N$ by

$$P_1 = (\cos\alpha/\sqrt{\cos 2\alpha})\,P_{(+)} - (\sin\alpha/\sqrt{\cos 2\alpha})\,2JN,$$

$$2N_1 = (\cos\alpha/\sqrt{\cos 2\alpha})\,2N + (\sin\alpha/\sqrt{\cos 2\alpha})\,JP_{(+)}.$$

Now we can proceed to the question of the effect of applying these matrix-tensors, which are invariant with respect to spinor transformations of the second type, to real spinors. Starting from Eqs. (10), (14), (15), (22) and (23), we get

$$F_{(+)}\psi_{(1)} = 2\,(\Omega_1 - \Omega_2 J)\,\psi_{(2)}. \tag{67}$$

Similarly,

$$F_{(+)}\psi_{(2)} = -2\,(\Omega_1 - \Omega_2 J)\,\psi_{(1)}. \tag{68}$$

Operating from the left on both sides of Eq. (67) with the matrix $P_{(+)}$ and using Eq. (66), we get

$$P_{(+)}\psi_{(2)} = 2N\psi_{(1)}. \tag{69}$$

and, similarly, applying the matrix N on the left and using Eq. (66) we find

$$P_{(+)}\psi_{(1)} = -2N\psi_{(2)}. \tag{70}$$

These same formulas could also have been obtained from Eq. (68). Formulas (67)-(70) are invariant with respect to spinor transformations of the second type.

Just as in the previous section, we may consider that the restriction to quantities invariant with respect to spinor transformations of the second type can be related to the fact that the quantities Ω_1, Ω_2, $P_{(+)}$ and N, and the matrix tensor $F_{(+)}$ found from them, are regarded as primary. If they are given, then the corresponding real spinors $\psi_{(1)}$ and $\psi_{(2)}$ will be determined to within a spinor transformation of the second type, by Eqs. (69) and (70).

In conclusion, we note that if together with $\psi_{(1)}$ and $\psi_{(2)}$ we consider the components of a vector A_α, which transform under a spinor transformation of the second type into $A_\alpha + (1/\epsilon)[\partial f/\partial x^\alpha]$, then the quantities

$$\psi^*_{(+)}R\frac{\partial\psi_{(+)}}{\partial x^\alpha} + 2\varepsilon\Omega_1 A_\alpha,$$

$$\psi^*_{(+)}R_4\frac{\partial\psi_{(+)}}{\partial x^\alpha} + 2\varepsilon\Omega_2 A_\alpha,$$

$$\psi^*_{(+)}R_4 R_\alpha\frac{\partial\psi_{(+)}}{\partial x^\beta} + i\varepsilon P_{(+)\alpha}A_\beta$$

$$\text{and}\quad \psi^*_{(+)}RR_\alpha\frac{\partial\psi_{(+)}}{\partial x^\beta} + 2\varepsilon n_\alpha A_\beta,$$

which are the components of a vector, pseudovector, second rank tensor, and second rank pseudotensor, respectively, will not change under such a transformation. Just as in the previous section, we could establish relations connecting them with one another and with other quantities, but we shall not take this up here.

Translated by M. Hamermesh
170