The Theory of the Acceleration of Charged Particles by Isotropic Gas Magnetic Turbulent Fields

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A theory of the acceleration of charged particles of an isotropic gas magnetic turbulence is developed. A kinetic equation is derived to determine the spectrum of rapid particles, and a method is given for solving this equation for various cases (stationary and non-stationary spectrum, account of magnetic-braking losses, etc.). The investigation makes wide use of the method of the theory of shower processes in cosmic rays.

THE considerable recent progress in the determination of the origin of cosmic rays is due to the rapid development of radio-astronomy[1,2]. It has become clear thereby that the most likely mechanism of cosmic-ray formation is the statistical acceleration of charged particles in a turbulent gas magnetic medium [2,3]. A certain role may also be played by the mechanism of induction acceleration in an increasing magnetic field[4,5]. An interesting variant of the statistical mechanism was proposed by Fermi[6], namely, the acceleration of particles in a "trap" between two approaching magnetic bursts such as may form in gas magnetic shock waves.

The present theories of statistical or induction acceleration[3,5] are not entirely satisfactory in view of their sketchiness, and in view of the incomplete account of the braking and absorption of particles. The question of the source of the primary acceleration of the particles, the so-called injection, still remains unsolved.

This article develops a theory of gas magnetic acceleration of charged particles, taking into account both the statistical and the induction mechanisms, and permitting us, at least in principle, to account for all possible conditions of injection, braking, and absorption of particles [7]. In the development of the theory we shall employ the mathematical methods of the theory of cosmic showers.

[1] I. S. Shklovskii, Astr. Zhurnal 30, 577 (1953)

[2] V. L. Ginzburg, Usp. Fiz. Nauk 5, 343 (1953)

[3] E. Fermi, Phys. Rev. 75, 1169 (1949)

[4] Ia. P. Terletskii, Usp. Fiz. Nauk 44, 46 (1951)

[5] A. A. Logunov and Ia. P. Terletskii, Izv. Akad. Nauk SSSR, Ser. Fiz. 17, 119(1953)

[6] E. Fermi, Astrophys. J. 119, 1 (1954)

[7] S. A. Kaplan, Circ. of L'vov Astronom. Obs. No. 27, 1953

1. ACCELERATION OF CHARGED PARTICLES BY GAS MAGNETIC FIELDS

1. Statistical Acceleration by Gas Magnetic Turbulence. This mechanism is discussed in many investigations[2,3,7,8]. We shall therefore cite here only the final equations. When particles "collide" with a magnetic field fluctuation that moves with a velocity u, the energy E of the particle (including the rest energy) changes by an amount

$$|\Delta E| \approx \frac{uv(E)}{c^2} E, \qquad (1)$$

here $v(E)$ is the velocity of the particle, and c is the velocity of light. The quantity ΔE is positive if u and v are in opposite directions, and negative otherwise. Thanks to the unequal probability of the positive and negative variations of the mean energy increment, we have

$$\overline{\Delta E} \approx \frac{u^2}{c^2} E. \qquad (2)$$

The particles are thus systematically accelerated in this case. The magnitude of this acceleration computed per unit length, is given by the following equation

$$\left(\frac{dE}{dx}\right)_{\text{turb}} = \left(\overline{\frac{u^2}{c^2 l}}\right) E = \alpha_1(t) E, \qquad (3)$$

where the averaging is performed over all vortex dimensions l that satisfy the conditions $l > r$ (inasmuch as Eqs. (1) and (2) are correct only in that case, in which the characteristic dimension of the fluctuation is greater than the radius of curvature r of the particle trajectory[7,8]). As the measure of the intensity of the energy fluctuations $D(E,t)$, namely the coefficient of diffusion along the "energy axis", we have

$$D(E, t) = \overline{(\Delta E)^2}/2l = \frac{1}{2c^2}\alpha_1(t)[Ev(E)]^2. \qquad (4)$$

[8] A. A. Logunov and Ia. P. Terletskii, J. Exper. Theoret. Phys. USSR 26, 129 (1954)

2. Induction Acceleration by Gas Magnetic Fields. It is well known that if the magnetic field intensity H increases systematically, the energy of the particles contained in that field also increases. To calculate the change in energy in this case it is most convenient to employ the adiabatic invariant[4]:

$$I = 3\pi c p_t^2 / eH. \tag{5}$$

Here p_t is the tangential component of the momentum and e is the charge of the particle. Assuming the velocities of the particles to be isotropic and using the well known relationship between the momentum and energy of the particles, we have

$$\frac{d \ln p_t^2}{dt} = \frac{d}{dt} \ln\left(\frac{E^2}{c^2} - m^2c^2\right) = \frac{d \ln H}{dt}.$$

Hence

$$\left(\frac{dE}{dx}\right)_{\text{ind}} = \frac{1}{v(E)}\left(\frac{dE}{dt}\right)_{\text{ind}} \tag{6}$$

$$= \frac{v(E)E}{2c^2}\frac{d \ln H}{dt} = \alpha_2(t)\frac{v(E)}{c}E.$$

3. Acceleration in Gas Magnetic Bursts. If a shock wave is formed in a gas located in a magnetic field, a sharp change in the field intensity and direction of the magnetic field takes place in the wave in addition to the discontinuity in density and pressure. If a particle is subjected to multiple "reflections" from a system of such bursts, the energy of the charged particle may increase by a noticeable fraction of its inertial value (reference 6 estimates this increase to be approximately 10-20%). It must be noted that the formation of a whole system of gas magnetic shock waves is quite frequently encountered in cosmic conditions. For example, the Crab Nebula, which is one of the most powerful sources of radio waves, is undoubtedly such a system, and is consequently also a source of cosmic radiation (according to references 1 and 2). However, this mechanism of acceleration[6] which, generally speaking, is a variant of the statistical mechanism, has not yet been studied in detail, and it is therefore still difficult to estimate its validity. If its role is really important, a very rough approximation can be obtained by using equations similar to (3) and (4), with additional factors, depending on multiplicity of the reflections in the "trap" – using Fermi's terminology[6]– i.e., in the region between two approaching bursts. For a more accurate analysis it is necessary to calculate the mean energy charge in the "trap" and then take into account the probability of the particle falling into this trap. For this, however, it is necessary to know the detailed structure of the system of gas-magnetic shock waves.

Although the acceleration by gas-magnetic shock waves is probably less important than the statistical or induction acceleration, this mechanism is apparently of primary importance for the injection, i.e., for the initial acceleration of the particles to a threshold that exceeds the level of energy losses by ionization. Let us first remark here that we detected intense sources of radio waves (testifying to the presence of rapid electrons) in the exact places where systems of gas magnetic shock waves exist (Crab Nebula, colliding galaxies, and similar objects[9]). Shklovskii[9] notes that when the shock waves have velocities of the same order as observed in the Crab Nebula, the thermal velocities of the electrons behind the wave front already approach the injection threshold. True, this does not explain the primary accelerations of the protons and of the heavy nuclei which, generally speaking, should be more probable than the acceleration of the electrons[2]. It seems to us that a thorough examination of the processes occurring in a gas-magnetic shock wave (including the process of the acceleration-in-a- "trap" type) will confirm the hypothesis concerning the injection of charged particles in gas magnetic shock waves. We shall not concern ourselves here with this question, partly because of its difficulty, and partly because the only thing we need to know for future considerations is the quantity $\nu(t)$, the total number of particles injected per unit volume per second. It is evident that

$$\nu(t) = \int_0^\infty \nu(E, t)\, dE, \tag{7}$$

where $\nu(E,t)$ is the number of injected particles with energies ranging from E to $E + dE$. The quantities $\nu(E,t)$ are obtainable by analysis of the acceleration process in gas magnetic shock waves. As far as $\nu(t)$ is concerned, we generally surmise that $\nu(t)$ is proportional to the energy dissipated in the gas magnetic shock waves. In the future this assumption will provide us with at least a qualitative estimate of the injection.

4. Expression for the Acceleration Parameters in Terms of Spectral Functions of Gas Magnetic Turbulence. If the gas magnetic medium in which the particles are accelerated is in a state of isotropic turbulence, the quantities u, l and H in Eqs. (3), (4) and (6) can be expressed in terms of

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[9] I. S. Shklovskii, Dokl. Akad. Nauk SSSR **98**, 353 (1954)

$F(k, t)$ and $M(k, t)$, namely, the spectral functions of the kinetic energy[10], respectively:

$$\alpha_1(t) = \frac{1}{\pi c^2}\int_0^\infty F(k, t)\, k\, dk, \tag{8}$$

$$\alpha_2(t) = \frac{1}{4c}\frac{d}{dt}\ln\int_0^\infty M(k, t)\, dk. \tag{9}$$

Taking into account the assumption that $\nu(t)$ is proportional to the energy dissipation in the gas magnetic shock waves, we can also write

$$\nu(t) \sim \int_0^\infty [F(k, t)\, k^3]^{1/2}\, [\zeta_f(k)\, F(k, t) - \zeta_m(k)\, M(k, t)]\, dk. \tag{10}$$

here the wave number $k = 2\pi/l$ and $\zeta_f(k)$ and $\zeta_m(k)$ are positive, slowly varying functions on the order of 1/10. The quantities $F(k,t)$ and $M(k,t)$ determine the kinetic and magnetic energy per unit mass, respectively, contained in vortices having wave numbers ranging from k to $k + dk$. For more details concerning the spectra of gas magnetic turbulence see reference 10.

2. BRAKING AND ABSORPTION OF PARTICLES

Along with the acceleration considered above, the moving particle will also experience braking because of the following processes:

(a) Ionization losses, as determined by the following equation[2]:

$$-\left(\frac{dE}{dx}\right)_{\text{ion}} = \frac{4\pi n_e e^4}{m_e v^2(E)}\, p(E, n_e) = \frac{\gamma c^2}{v^2(E)}, \tag{11}$$

where p is a function that exhibits a weak dependence on its arguments, and ranging approximately from 20 to 200 under cosmic conditions[2].

(b) Braking of rapid electrons by radiation in the magnetic field.

$$-\left(\frac{dE}{dx}\right)_{\text{mag}} = \frac{2}{3}\left(\frac{e}{m_e c^2}\right)^4 H_\perp^2 E^2 = \beta(t)\, E^2,$$
$$\beta(t) = \frac{16\pi\rho}{9}\int_0^\infty M(k, t)\, dk, \tag{12}$$

where ρ is the gas density. Here it is assumed $\overline{H_\perp^2} = 2\overline{H^2}/3$.

(c) Braking of electrons by photon radiation and braking of protons by meson radiation. We shall denote by $(1/x_0)\, \varphi_0(y)\, dy$, where x_0 is the radiation unit length, the differential probability that a particle with energy E will have an energy in the interval from $(1 - y)E$ to $(1 - y - dy)E$ after radiating a photon or a meson over a one centimeter path. For the braking of electrons in un-ionized hydrogen we have:

$$\frac{1}{x_0} = \frac{4n}{137}\ln(137)\left(\frac{e^2}{mc^2}\right)^2. \tag{13}$$

If the hydrogen is ionized, x_0 is several times greater. The value of x_0 for the braking of protons by meson radiation is on the order of 13 (this is a coincidence). In the case of total shielding, the function $\varphi_0(y)\, dy$ has the following form[11]:

$$\varphi_0(y)\, dy = \left[1 + y^2 - \frac{2}{3}\, y\right] dy/y. \tag{13'}$$

Usually the acceleration particles occur in regions where the gas is ionized and where the shielding consequently takes place only in the case of very large energies. In the absence of shielding it is necessary to multiply (13′) by a factor that is logarithmically dependent on the energy. But this factor, which complicates the theory excessively, does not appear to be of importance in the problems discussed by us. So far, we know of no equations analogous to (13′) for the braking of protons by meson radiation.

(d) Energy pulses by electron due to the inverse Campton effect[12]:

$$-\left(\frac{dE}{dx}\right)_{\text{Comp}} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \bar{n}_{\text{ph}}\bar{\varepsilon}_{\text{ph}}\left(\frac{E}{m_e c^2}\right)^2 \tag{14}$$

Here $\bar{n}_{\text{ph}}$ and $\bar{\varepsilon}_{\text{ph}}$ are the average number and average energy of the scattering photons. This mechanism is thus analogous in its energy dependence to the braking of electrons in a magnetic field, differing from it only in that in this case the energy losses occur, so to speak, in large doses.

(e) Finally, in some cases it is also necessary to take into account the "annihilation" of fast particles, for example, in the case of nuclear reactions. We shall denote the length of the mean free path for this process by Λ. Evidently

$$1/\Lambda = \sum_A n_A \sigma_A, \tag{15}$$

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[10] S. A. Kaplan, Dokl. Akad. Nauk SSSR **94**, 33 (1954); J. Exper. Theoret. Phys. USSR **27**, 699 (1954)

[11] S. Z. Belen'kii, *Shower Processes in Cosmic Rays*, GITTL, Moscow, 1948

[12] L. D. Landau and Iu. B. Rumer, Proc. Roy. Soc. (London) **A166**, 213 (1938)

where n_A is the number of nuclei capable of entering into reaction per unit volume and σ_A is the corresponding effective cross section.

The first two energy-loss mechanisms are continuous. We shall take account of them by writing the algebraic sum for the value of the systematic change in energy.

$$\left(\frac{dE}{dx}\right)_{\text{tot}} = \left(\frac{dE}{dx}\right)_{\text{turb}} + \left(\frac{dE}{dx}\right)_{\text{ind}} + \left(\frac{dE}{dx}\right)_{\text{mag}} + \left(\frac{dE}{dx}\right)_{\text{ion}} \quad (16)$$

The remaining loss mechanisms should be accounted for by the corresponding terms in the kinetic equation. The most difficult to account for is the inverse Compton effect, because of its nonlinear dependence on the energy. In those (relatively rare) cases, when this effect must be taken into account, we shall therefore simply add Eq. (14) to Eq. (16).

In concluding this section, let us note that along with the acceleration, absorption, and braking of particles, it is also necessary in certain cases to take into account the direct formation of fast particles, for example, in the decomposition of mesons into electrons, in pair formations, in nuclear reactions, etc. These processes can be accounted for either by adding corresponding terms in the expressions for the injection, or else by setting up a system of kinetic equations with terms that account for the production of particles of a given kind by particles of another kind.

3. KINETIC EQUATION FOR THE SPECTRUM OF RAPID PARTICLES

In many cases of practical interest we can assume that the particle accelerates in the region of isotropic and homogeneous gas magnetic turbulence, where the primary sources, the injectors, are also uniformly distributed. This condition is satisfied, for example, if the injectors are gas-magnetic shock waves, which are bound to occur in a region occupied by a supersonic gas magnetic turbulence. In these cases we can disregard the spatial diffusion of particles, greatly simplifying the subsequent computations. Let us remark, however, that accounting for the spatial diffusion does not lead to any difficulties, in principle,in the mathematics[5] (see below).

In formulating the kinetic equation for the distribution function $N(E,t)$ of rapid electrons we must take into account both the systematic and stochastic accelerations of the particles (section 1), as well as the energy losses and absorption of the particles (section 2), and finally the injection. The systematic and stochastic acceleration, as well as the "continuous" losses, are taken into account using a method that is customary in this theory, that is, by writing down the expression[5]:

$$-\frac{\partial}{\partial E}\left[\left(\frac{dE}{dx}\right)_{\text{tot}} N(E,\ t)\right] + \frac{\partial^2}{\partial E^2}\left[D(E,\ t)\, N(E,\ t)\right]. \quad (17)$$

The braking by the radiation is taken into account in accordance with the theory of cosmic showers, using the following expression[11]:

$$\frac{1}{x_0}\int_0^1\left[\frac{1}{1-y}N\left(\frac{E}{1-y},\ t\right) - N(E,\ t)\right]\varphi_0(y)\,dy. \quad (18)$$

The absorption of particles is accounted for by a term $\frac{1}{\Lambda}N(E,t)$. We thus have for the total change in the number of particles per unit volume per second

$$\frac{1}{v(E)}\frac{\partial N(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left[\left(\frac{dE}{dx}\right)_{\text{tot}} N(E,\ t)\right] + \frac{\partial^2}{\partial E^2}\left[D(E,\ t)\, N(E,\ t)\right]$$
$$+\frac{1}{x_0}\int_0^1\left[\frac{1}{1-y}N\left(\frac{E}{1-y},\ t\right) - N(E,\ t)\right]\varphi_0(y)\,dy - \frac{1}{\Lambda}N(E,\ t) + \frac{\nu(E,\ t)}{v(E)} \quad (19)$$

This equation, together with Eqs. (3), (4), (6), (8) to (13), (15), and (16), determines the spectrum of the rapid particles. When solving this equation it is necessary to assume that the coefficients of this equation, together with the expression for the injection and the initial distribution function of the particles, areknown.

Equation (19) is valid for both relativistic and nonrelativistic regions of the particle spectrum. From now on, we shall restrict ourselves to the simpler and at the same time more interesting relativistic particles. In fact, if we exclude the corpuscular streams from the sun, where the acceleration is probably of more complicated character than the acceleration of the isotropic gas magnetic turbulence considered here, we need consider in all the remaining cases only the relativistic region of the spectrum, both with respect to cosmic rays directly, as well as with respect to the magnetic-braking radiation. Besides, all the simplifications are reduced in the relativistic case to the assumption $v(E) = c$. The results obtained below will, therefore, be valid, in rough approximation, for nonrelativistic particles that are not too slow.

Thus, assuming $v(E) = c$, and taking into account the expressions given above for the coefficients of Eq. (19), we obtain

$$\frac{\partial N(E,t)}{dt} = -c\frac{\partial}{\partial E}\left[\left(\alpha(t)E - \beta(t)E^2 - \gamma\right)N(E,t)\right] + \frac{1}{2}c\alpha_1(t)\frac{\partial^2}{\partial E^2}[E^2N(E,t)] + \frac{c}{x_0}\int_0^1\left[\frac{1}{1-y}N\left(\frac{E}{1-y},t\right) - N(E,t)\right]\varphi_0(y)\,dy - \frac{c}{\Lambda}N(E,t) + \nu(E,t). \quad (20)$$

Here $\alpha(t) = \alpha_1(t) + \alpha_2(t)$, where we shall also include the acceleration in the "traps". The solution of Eq. (20) will be discussed below

Let us remark again that Eq. (20), like Eq. (19), does not take into account the spatial diffusion of the charged particles, occurring when the injector distribution is non-uniform. If it is necessary to take this into account, a term $lc\Delta_r N(E,t,r)/3$ is added to the first part of Eq. (20), where Δ_r is the Laplacian and the spectrum $N(E,t,r)$ depends also on the coordinate r.

Equations of the type (20), taking into account the spatial diffusion, are discussed in reference 5, where it is assumed, however, that the acceleration of the particles was due to an induction mechanism that becomes operative when the magnetic field intensity increases. In addition, reference 5 does not account for the braking with sufficient accuracy (the third term of the right half of (20) is lacking). Instead of taking into account the last term of (20), reference 5 seeks a source-type of solution, i.e., all particles are assumed to "escape" into the accelerating medium with the same energy. Thus Eq. (20) (see also reference 7) is more general than the analogous equation in reference 5 (in the sense that the braking and the acceleration of the particles are accounted for), and as we shall see below, it permits finding a more general solution, one that approximates more closely the actual cosmic conditions.

In conclusion let us remark, that if we are not interested in the dependence of the particle spectrum on the coordinates it is easy to calculate the particle diffusion from the region of the gas-magnetic medium consideration, by adding to the right half of (20) a term $(2cl/L^2)N(E,t)$, where L represents the linear dimensions of the system. In other words, instead of Λ it is now necessary to substitute in Eq. (20) the quantity

$$\frac{1}{\Lambda} = \sum n_A\sigma_A + \frac{2\bar{l}}{L^2}. \quad (15')$$

Equation (20), together with (15′), accounts for the diffusion of particles from the acceleration region with sufficient accuracy for the assumptions of the theory. Generally speaking, as follows from the observed fact that the proton spectra are similar to those of heavy particles, the first term of (15′) is smaller than the second term[6].

4. SOLUTION OF KINETIC EQUATION

Equation (20), the same as the kinetic equations of the theory of cosmic showers, is best investigated with the aid of the Mellin transform[12]. Multiplying (20) by $E^s\,dE$, integrating from 0 to ∞, and assuming that the distribution function vanishes both at $E = 0$ as well as at $E \to \infty$, we obtain

$$\partial\mathfrak{N}(s,t)/\partial t = {}^1/_2 c[s(s+1)\alpha_1(t) + 2s\alpha_2(t)]\mathfrak{N}(s,t) - cs\beta(t)\mathfrak{N}(s+1,t) - c\gamma\mathfrak{N}(s-1,t) - \frac{c}{x_0}A(s)\mathfrak{N}(s,t) - \frac{c}{\Lambda}\mathfrak{N}(s,t) + \int_0^\infty E^s\nu(E,t)\,dE. \quad (21)$$

The following designations are used here:

$$\mathfrak{N}(s,t) = \int_0^\infty E^s N(E,t)\,dE \quad (22)$$

and

$$A(s) = \int_0^1 [1-(1-y)^s]\,\varphi_0(y)\,dy. \quad (23)$$

The function $A(s)$ is tabulated in reference 11 for electron braking by photon radiation. The quantity s, which becomes the variable in Eq. (21) instead of the variable E in Eq. (20), can be complex. After finding the function $\mathfrak{N}(s,t)$ from (21), the particle spectrum is determined with the inverse Mellin transform:

$$N(E,t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}E^{-s-1}\mathfrak{N}(s,t)\,ds, \tag{24}$$

where the path of integration is a straight line parallel to the imaginary axis in the positive half of the complex plane[11].

Equation (21) is a differential equation in t and a finite difference equation in s. The general solution of (21) will be discussed below, and we shall restrict ourselves at the present time to the case when the terms with $\beta(t)$ and γ can be neglected. Then (21) can be integrated with respect to time directly

$$\mathfrak{N}(s,t)=\int_0^t \nu(s,t')\exp\{I(s,t,t')\}dt'. \tag{25}$$

where the following designations are introduced:

$$\nu(s,t)=\int_0^\infty \nu(E,t)E^s dE, \tag{26}$$

$$\Phi(s,t)=\frac{1}{2}c\,[s(s+1)\alpha_1(t)+2s\alpha_2(t)] - c\left[\frac{A(s)}{x_0}+\frac{1}{\Lambda}\right], \tag{27}$$

$$I(s,t,t')=\int_{t'}^{t}\Phi(s,t'')\,dt''.$$

In addition, we subject (25) to an initial condition $\mathfrak{N}(s,0)=0$, which does not limit the physical generality of the solution. The particle spectrum is obtained, as was already noted, but inserting (25) into (24)

$$N(E,t)=\frac{1}{2\pi iE}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E^s}\int_0^t dt'\nu(s,t')\exp[I(s,t,t')]. \tag{28}$$

Expression (28) is calculated by the method of steepest descents, as is usually done in the theory of cosmic showers. This equation can be rewritten in the following form

$$N(E,t)=\frac{1}{2\pi iE}\int_0^t dt'\int_{\delta-i\infty}^{\delta+i\infty}\frac{\nu(s,t')}{(mc^2)^s}\times\exp\left\{-s\ln\frac{E}{mc^2}+I(s,t,t')\right\}ds. \tag{29}$$

The factor before the exponential in (29) is slowly-varying in s. As s varies along the real axis, the exponential index reaches a minimum at the point $s=s_m(I_{t'})$ under the following condition

$$\ln\frac{E}{mc^2}=\int_{t'}^{t}\left(\frac{\partial\Phi(s,t'')}{\partial s}\right)_{s=s_m}dt'', \tag{30}$$

and consequently the integrand has a maximum at these points as s varies along the imaginary axis. Continuing with the usual procedure of the method of steepest descents[11], we obtain

$$N(E,t)=\frac{1}{E}\int_0^t\frac{\nu(s_m,t')}{(mc^2)^{s_m}}\left[\exp\left\{-s_m\ln\frac{E}{mc^2}+I(s_m,t,t')\right\}\right]\Bigg/\sqrt{2\pi\left|\int_{t'}^{t}\left(\frac{\partial^2\Phi}{\partial s^2}\right)_{s=s_m}\right|dt''} \tag{31}$$

Further integration with respect to time depends naturally on the actual form of the dependence of ν and α on t. In this integration it is necessary to bear in mind that the value of s_m also depends on time as in Eq. (30).

Let us note that if nonrelativistic particles are injected (for example in the case of gas magnetic shock waves), we have

$$\nu(s,t)\approx(mc^2)^s\int_0^\infty\nu(E,t)\,dE=(mc^2)^s\nu(t) \tag{32}$$

in accordance with (7).

Furthermore, it follows from (22) that $\mathfrak{N}(0,t)$ is the total number of high speed particles per unit volume, and $\mathfrak{N}(1,t)$ is the total energy of the rapid particles. Consequently, Eq. (25) at $s=0$ and at $s=1$ determines the time dependence of the total number of particles per unit volume and of their total energy per unit volume, respectively. Let us apply the solution of (25) and (31) to the case of a stationary turbulence, that is, let us assume that α and ν are independent of time. Then

$$\mathfrak{N}(s,t)=\nu(mc^2)^s(e^{\Phi(s)t}-1)/\Phi(s), \tag{33}$$

$$N(E,t) = \frac{\nu}{E}\int_0^t \left[\exp\left\{-s_m \ln\frac{E}{mc^2} + \Phi(s_m)(t-t')\right\} \Big/ \sqrt{2\pi\left|\frac{d^2\Phi}{ds^2}\right|_{s=s_m}(t-t')}\right] dt', \tag{34}$$

where s_m is determined from the condition:

$$\ln = \left(\frac{d\Phi}{ds}\right)_{s=s_m}(t-t'). \tag{35}$$

and.

$$\Phi(s) = \frac{1}{2}c\left[s(s+1)\alpha_1 + 2s\alpha_2\right] - c\left[\frac{A(s)}{x_0} + \frac{1}{\Lambda}\right]. \tag{36}$$

For large values of t, integral (34) can also be evaluated by the method of steepest descents. If we ignore the dependence of the factor in front the exponent in (34) on t', the exponent reaches a maximum at

$$\frac{ds_m}{dt'}\left\{-\ln\frac{E}{mc^2} + \left(\frac{d\Phi}{ds}\right)_{s=s_m}(t-t')\right\} - \Phi(s_m) = 0, \tag{37}$$

hence, according to (35), $\Phi(s_m) = 0$. After simple calculation we obtain finally

$$N(E)\,dE = \frac{\nu}{\left|\frac{d\Phi}{ds}\right|}\left(\frac{mc^2}{E}\right)^{s_0+1}\frac{dE}{mc^2}, \tag{38}$$

where s_0 is the root of equation $\Phi(s) = 0$. If this equation has two positive nonvanishing roots, it is necessary to take the smaller of the two, for in view of the statistical character of the acceleration mechanism, the system comprising the gas magnetic medium and rapid particles will tend toward equipartition of the energy, that is, to a more sloping spectrum of rapid particles.

It also follows from (35) that the time $\tau(E)$ required to establish an equilibrium spectrum is on the order of

$$(E) \approx \frac{1}{\left|\frac{d\Phi}{ds}\right|_{s_0}}\ln\frac{E}{mc^2}. \tag{39}$$

The spectrum (38) occurs also in the nonstationary turbulence state, provided the characteristic time of its variation exceeds $\tau(E)$ from (39).

By way of illustration let us give the following example. By reference 2, the values of $c\alpha$ and c/Λ in interstellar space can be assumed to be

$$c\alpha_1 = 5\times10^{-17}\ \text{sec}^{-1},$$

$$c\alpha_2 = 0,\ c/\Lambda = cn\sigma = 3\times10^{10}\times0.1\times2.5\times10^{-26}.$$

Substituting these values into (36) and assuming $\Phi(s_0) = 0$, we obtain $s_0 = 1.3$. Substituting into (38) we get:

$$N(E)\,dE = \nu\cdot10^{16}\left(\frac{10^9}{E}\right)^{2.3}\frac{dE}{10^9},$$
$$N(E>E_0) = \nu\,\frac{10^{16}}{1.3}\left(\frac{10^9}{E}\right)^{1.3}. \tag{40}$$

Here the energy is given in electron volts. According to references 1 and 2, the sources of cosmic radiation are supernova bursts. Considering them to be injectors, we have as an estimate for

$$\nu = \frac{\text{number of rapid particles formed in supernova}}{\text{frequency of supernova burst}\times\text{volume of galaxy}}$$

$$\approx \frac{10^{51}}{30\times3\times10^7\times10^{68}} \approx 10^{-26}\text{sec}^{-1}\ \text{cm}^{-3}. \tag{41}$$

Substituting into (40) we get

$$N(E)\,dE \approx 10^{-10}\left(\frac{10^9}{E}\right)^{2.3}\frac{dE}{10^9},$$

$$N(E>1.5\cdot10^9) \approx 5\cdot10^{-11}\ \text{cm}^{-3}$$

which is in rather good agreement with observed data.

We gave this simple numerical calculation only as an example, without dwelling at all on the great importance of particular acceleration in interstellar space. It is also possible to consider in an analogous manner the acceleration of particles in the expanding ejected supernova shells. Inasmuch as we deal here with a nonstationary problem the starting equation is (31).

In conclusion let us dwell shortly on the calculation of the ionization and magnetic-braking losses. Inasmuch as the former occur at low particle energies, and the latter at large energies, there is no practical interest in joint evaluation of both effects. We shall restrict ourselves here only to a treatment of the method of computing the magnetic-braking losses. The ionization losses are evaluated in a similar manner. Let us note here that both losses will be evaluated by a method developed by Belen'kii[11] for the calculation of ionization losses in the theory of cosmic showers.

It is difficult to obtain a general solution for (21) when $\beta(t) \neq 0$. It is much easier to solve the problem in that case when the turbulence can be assumed stationary, that is, when α, β, and ν are independent of time. We shall restrict ourselves to this case.

Taking the Laplace transform of (21) at $\gamma = 0$, we obtain

$$[\Phi(s) - \lambda] R(s, \lambda) - c\beta s R(s+1, \lambda) + \nu (mc^2)^s/\lambda = 0, \quad (42)$$

where

$$R(s, \lambda) = \int_0^\infty e^{-\lambda t} \mathfrak{R}(s, t)\, dt. \quad (43)$$

Writing analogous equations for $R(S+1, \lambda)$, $R(s+2, \lambda)$, etc. and using successive elimination, we obtain

$$R(s, \lambda) = -\frac{\nu}{\lambda[\Phi(s) - \lambda]} \sum_{k=0}^{\infty} \frac{(-\beta mc^3)^k (s+1)(s+2)\ldots(s+k+1)}{[\Phi(s+1) - \lambda][\Phi(s+2) - \lambda]\ldots[\Phi(s+k+1) - \lambda]}. \quad (44)$$

Equation (44) is the solution to the problem. The transformation from $R(s, \lambda)$ to $N(E, t)$ is performed with the aid of the inverse Mellin and Laplace transform[11]:

$$N(E, t) = -\frac{1}{4\pi} \int_{\delta - i\infty}^{\delta + i\infty} ds \int_{d - i\infty}^{d + i\infty} d\lambda R(s, \lambda) E^{-s-1} e^{\lambda t}. \quad (45)$$

The series (44) is essentially an expression of $R(s, \lambda)$ in powers of $\beta mc^2/\alpha$, i.e., this series should converge quite rapidly. It is therefore necessary to retain only one or two terms when substituting (44) into (45). In analogy with reference 11, it is possible to simplify the calculation by using the following approximation: $\Phi(s) - \lambda = f(\lambda)[s - s_1(\lambda)][s - s_2(\lambda)]$. This approximation is accurate when $x_0 \to \infty$. Here s_1 and s_2 are the roots of the equation $\Phi(s) - \lambda = 0$.

Finally, taking only the inverse Laplace transform of (44) and assuming $s = 0$ or $s = 1$, it is possible to obtain the total number of particles and their total energy per unit volume respectively. We shall not go through all these computations here.

It follows therefore from all that has been said above that in many cases of practical interest we can calculate the principal characteristics of the particle spectrum of particles accelerated by a gas magnetic turbulence to a sufficient degree of accuracy using relatively simple mathematical methods. The methods of these computations account for the various types of possible injectors and lead to sufficiently general assumptions concerning the gas magnetic turbulence that accelerates the particle.

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