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The Asymptotic Green's Function of Nucleon and Meson in Pseudo-Scalar Theory with Weak Interaction

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The asymptotic Green's function (for $|p^2| \gg m^2$) of nucleon and meson is examined in pseudoscalar theory with pseudoscalar coupling for small values of the coupling constant. The starting point is a set of equations proposed in reference 1. In contradistinction to this work, the renormalization of mass and charge are performed according to a method developed in reference 2, and it is proved that that method removes all infinities from the problem.

1. FORMULATION OF THE EQUATIONS

In reference 2, covariant equations were obtained which described a single nucleon interacting with a pseudoscalar meson field, and the renormalization of mass and charge was performed in them. In the present work, we apply that method of mass and charge renormalization in a problem of some methodological interest, that of the asymptotic behavior (for $|p^2| \gg m^2$) of the Green's functions $G(\mathbf{p})$ of the nucleon and $D(p^2)$ of the meson* for small g^2. In this case, it will be seen that the factor Z_1 (remaining in the equations of reference 2), which can be infinite, is automatically excluded and does not appear in the result. Thus, the method of removing infinities whose definition does not involve perturbation theory will be illustrated on a concrete example.

For the study of the asymptotic Green's function it is necessary to select the dominant terms from the infinite system of "branching" equations. From perturbation theory, it is known that the diagrams for $G(\mathbf{p})$ and $D(p^2)$ behave as $g^{2n} \times [\ln (p^2/m^2)]^m$ where $m \leq n$ [3,4]. For $g^2 \ll 1$, diagrams with $n = m$ dominate. As was proved in reference 1, the condition $n = m$ makes it possible to reduce the infinite system of equations to three equations for the three functions $G(\mathbf{p})$, $D(p^2)$ and $\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$. The function $\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$ actually depends on two momenta (for example, on the initial momentum $\mathbf{p}$ and final momentum $\mathbf{p}-\mathbf{k}$ of the nucleon). For convenience in the following calculation, we write three momenta, the redundant meson momentum being separated by a semi-colon.

In pseudoscalar symmetrical theory, these functions are subject to the following system of equations[1]:

$$\Big\{\mathbf{p} - m_0 - 3\frac{g_0^2}{4\pi^3 i}\gamma_5 \int d^4k G(\mathbf{p}-\mathbf{k}) \times \Gamma_5(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k}) D(k^2)\Big\} G(\mathbf{p}) = 1; \quad (1)$$

$$\Big\{k^2 - \mu_0^2 + 2\frac{g_0^2}{4\pi^3 i}\int d^4p \,\mathrm{Sp}\,\gamma_5 G(\mathbf{p}) \times \Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) G(\mathbf{p}+\mathbf{k})\Big\} D(k^2) = 1; \quad (2)$$

$$\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k}) = \gamma_5 - \frac{g_0^2}{4\pi^3 i}\int \Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{q}; -\mathbf{q}) G(\mathbf{p}-\mathbf{q}) \times \Gamma_5(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}-\mathbf{k}; -\mathbf{k}) G(\mathbf{p}-\mathbf{q}-\mathbf{k}) \times \Gamma_5(\mathbf{p}-\mathbf{q}-\mathbf{k}, \mathbf{p}-\mathbf{k}; \mathbf{q}) D(q^2) d^4q. \quad (3)$$

We clarify the correspondence between these equations and the system of branching equations of reference 2. Equations (1) and (2) coincide* with Eqs. (5) II and (7) II if one substitutes there the expressions for $M_0(\mathbf{p})$ and $P_0(k^2)$ from Eqs. (9) II and (10), II, respectively, and notes that $\Gamma_5 = \gamma_5 + M_1(\mathbf{p}, \mathbf{k})$. If in Eqs. (5) – (10) II, we set $M_n = 0$ for $n \geq 2$ and $P_n = 0$ for $n \geq 1$, then from

* In reference 2 these functions were denoted by $G_0(p)$, $D_0(p^2)$; the remaining notation agrees with that paper. References to the formulas of reference 2 will be denoted by II.

* Symmetrical pseudoscalar theory differs from neutral pseudoscalar theory in the appearance in Eq. (1) of the coefficient 3 in front of the integral, and the appearance in Eqs. (2) and (3) of the corresponding coefficients 2 and – 1.

[1] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 497, 773, 1177 (1954); 96, 261 (1954)

[2] A. D. Galanin, B. L. Ioffe and I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR 98, 361 (1954)

[3] A. I. Akhiezer and V. B. Beresetskii, *Quantum Electrodynamics*, Moscow, 1953, p. 210

[4] A. D. Galanin, J. Exper. Theoret. Phys. USSR 27, 417 (1954)

Eqs. (6) II and (9) II we obtain an equation differing somewhat from Eq. (3):

$$\Gamma_5(\mathbf{p},\ \mathbf{p}-\mathbf{k};\ -\mathbf{k}) = \gamma_5 - \frac{g_0^2}{4\pi^3 i}\gamma_5 \int G(\mathbf{p}-\mathbf{q})\Gamma_5(\mathbf{p}-\mathbf{q},\ \mathbf{p}-\mathbf{q}-\mathbf{k};\ -\mathbf{k}) \times G(\mathbf{p}-\mathbf{q}-\mathbf{k})\Gamma_5(\mathbf{p}-\mathbf{q}-\mathbf{k},\ \mathbf{p}-\mathbf{k};\ \mathbf{q})D(q^2)\,d^4q. \qquad (4)$$

If we solve the system of Eqs. (1), (2) and (4) by iteration, then we find that not all diagrams are obtained of the class which interests us. Namely, diagrams will be absent which are obtained by inserting a vertex part in the left-most vertex of the basic diagram given in Fig. 1 for Γ_5 (see, for example, the diagram depicted in Fig. 2).

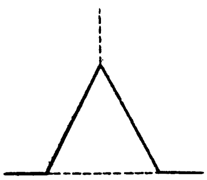

FIG. 1

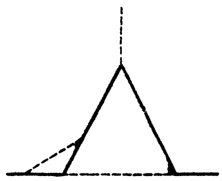

FIG. 2

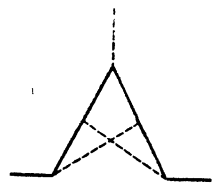

FIG. 3

Now, each diagram in which every integration over an interior meson line leads to an infinity in the unrenormalized theory will belong to the class which interests us. In fact, after renormalization, each infinity is replaced by $\ln(p^2/m^2)$ and asymptotically such a diagram will behave as $g^{2n}[\ln(p^2/m^2)]^n$, i.e., such that we ought to take account of it. In particular, the diagram depicted in Fig. 2 must properly be taken into account, since, asymptotically, it has the form

$$g^4\left(\ln\frac{p^2}{m^2}\right)^2.$$

Therefore, in Eq. (9) II for M_n with $n \geq 2$, one cannot set $M_n = 0$, but rather must take into account the set of those terms which correspond to the insertion of a vertex part in the left-most vertex of Fig. 1. Here it is necessary to select only terms in which each integration over an internal meson line leads to infinities, and not to take into account "overlapping" diagrams [for example, look at Fig. 3 which behaves asymptotically as $g^4\ln(p^2/m^2)$]. Proceeding in the indicated fashion, we pass from Eq. (4) to Eq. (3).

We now perform the renormalization of mass and charge in Eqs. (1) – (3). Since Eqs. (1) and (2) coincide with the corresponding equations of II, their renormalization can be performed just as in II. Therefore, it remains only to renormalize Eq. (3). Following the method developed in II, we introduce in place of $\Gamma_5(\mathbf{p},\mathbf{p}-\mathbf{k};-\mathbf{k})$, the renormalized Green's function $\Gamma_5^*(\mathbf{p},\mathbf{p}-\mathbf{k};-\mathbf{k}) = Z_1\Gamma_5(\mathbf{p},\mathbf{p}-\mathbf{k};-\mathbf{k})$ where $Z_1^{-1} = \gamma_5\Gamma_5(m,m,\mu)$ and m and μ are the experimental masses of nucleon and meson, respectively. We use the following connection between the renormalized and unrenormalized Green's functions of nucleon and meson [compare Eq. (22) II] $G^* = Z_2^{-1}G$, $D^* = Z_3^{-1}D$, where

$$Z_2 = 1 + M_0'^*(m);\ Z_3 = 1 + P_0'^*(\mu^2), \qquad (5)$$

and the experimental and fictive charge, g and g_0, respectively, are related by: $g_0^2 = Z_1^2 Z_2^{-2} Z_3^{-1} g^2$. Then, in place of Eq. (3), we get the following equation for the renormalization function $\Gamma_5^*(\mathbf{p},\mathbf{p}-\mathbf{k};-\mathbf{k})$:

$$\Gamma_5^*(\mathbf{p},\ \mathbf{p}-\mathbf{k};\ -\mathbf{k}) = Z_1\gamma_5 - \frac{g^2}{4\pi^3 i}\int\Gamma_5^*(\mathbf{p},\ \mathbf{p}-\mathbf{q};\ -\mathbf{q})G^*(\mathbf{p}-\mathbf{q}) \times \Gamma_5^*(\mathbf{p}-\mathbf{q},\ \mathbf{p}-\mathbf{q}-\mathbf{k};-\mathbf{k})G^*(\mathbf{p}-\mathbf{q}-\mathbf{k}) \times \Gamma_5^*(\mathbf{p}-\mathbf{q}-\mathbf{k},\ \mathbf{p}-\mathbf{k};\ \mathbf{q})D^*(q^2)\,d^4q. \qquad (6)$$

Equation (6) can be written conveniently in another form, if Z_1 is expressed in terms of the renormalized function and experimental charge:

$$Z_1^{-1} = 1 + \gamma_5 M_1(m,\ \mu) = 1 + \gamma_5 Z_1^{-1} M_1^*(m,\ \mu), \qquad (7)$$

because $M_1^* = Z_1 M_1$. Thus,

$$Z_1 = 1 - \gamma_5 M_1^*(m, \mu) \qquad (7')$$

and Eq. (6) takes the following form:

$$\Gamma_5^*(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k}) = \gamma_5 + M_1^*(\mathbf{p}, \mathbf{k}) - M_1^*(m, \mu), \qquad (8)$$

where

$$M_1^*(\mathbf{p}, \mathbf{k}) = -\frac{g^2}{4\pi^3 i}\int \Gamma_5^*(\mathbf{p}, \mathbf{p}-\mathbf{q}; -\mathbf{q}) \times G^*(\mathbf{p}-\mathbf{q})\Gamma_5^*(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}-\mathbf{k}; -\mathbf{k}) \times G^*(\mathbf{p}-\mathbf{q}-\mathbf{k}) \times \Gamma_5^*(\mathbf{p}-\mathbf{q}-\mathbf{k}, \mathbf{p}-\mathbf{k}; \mathbf{q}) D^*(q^2)\, d^4q. \qquad (9)$$

Equations (8) and (9), together with the renormalized Eq. (23) II for $G^*(\mathbf{p})$ and $D^*(k^2)$:

$$\{\mathbf{p} - m - [M_0^*(\mathbf{p}) - M_0^*(m) - (\mathbf{p}-m) M_0^{*\prime}(m)]\}\, G^*(\mathbf{p}) = 1; \qquad (10)$$

$$M_0^*(\mathbf{p}) = 3\frac{g^2}{4\pi^3 i} Z_1 \gamma_5 \int G^*(\mathbf{p}-\mathbf{k})\Gamma_5^*(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k}) D(k^2)\, d^4k; \qquad (11)$$

$$\{k^2 - \mu^2 - [P_0^*(k^2) - P_0^*(\mu^2) - (k^2-\mu^2) P_0^{\prime *}(\mu^2)]\}\, D(k^2) = 1; \qquad (12)$$

$$P_0^*(k^2) = -2\frac{g^2}{4\pi^3 i} Z_1 \operatorname{Sp} \gamma_5 \int G^*(\mathbf{p})\Gamma_5^*(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) G(\mathbf{p}+\mathbf{k})\, d^4p \qquad (13)$$

and Eq. (7′) for Z_1 constitute the complete system for the determination of the asymptotic behavior of the functions G^* and D^*.

2. EQUATIONS FOR THE GREEN'S FUNCTION OF THE NUCLEON

Since we are interested only in the asymptotic function G, D and Γ_5[†], we can neglect m^2 and k^2. Therefore, a solution of Eqs. (7) - (13) will be of the form[††]

$$G(\mathbf{p}) = \mathbf{p}^{-1}F(p^2), \quad D(k^2) = k^{-2}\varphi(k^2), \qquad (14)$$

where F and ϕ are slowly varying (for example, logarithmic) functions of their arguments.

The general form of the function $\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$, when its arguments are large compared to m and $|k^2| \gg |p^2|$, is as follows:

$$\Gamma_5(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k}) = \gamma_5\left\{s_0[(\mathbf{p}-\mathbf{k})^2, p^2, k^2] - \frac{\mathbf{kp}}{k^2+p^2} s_1[(\mathbf{p}-\mathbf{k})^2, p^2, k^2]\right\}. \qquad (15)$$

The function $s_0[(\mathbf{p}-\mathbf{k})^2, p^2, k^2]$ is a slowly varying function of its arguments when written in terms of the independent variables (this form is suggested by perturbation theory, in which it is easy to write down s_0 in first approximation and verify the following argument).

For $|k^2| \gg |p^2|$, the second term of Eq. (15) is very much smaller than the first; however, it is necessary to take it into account, since, when it is substituted in the mass operator (11), we still get a logarithmically divergent integral. This is connected with the fact that s_1 (as can be shown in perturbation theory in corroboration of the following computation), considered as a function of two variables $s_1 = s_1(p^2, k^2)$, grows as $\ln(k^2/p^2)$ for $|k^2| \gg |p^2|$ but for $|k^2| \sim |p^2|$ it approaches a constant, which we shall neglect.

At first glance, it might seem that momenta k^2 of order p^2 should be significant in the integration over four dimensional k-space which occurs in the renormalized mass operator appearing in Eq. (10). That is correct, however, only for the first term, $\gamma_5 s_0$, of Γ_5 appearing in Eq. (15). Indeed, from perturbation theory it is known that the subtraction from $M_0(\mathbf{p})$ of the expressions $M_0(m)$ and $(\mathbf{p}-m)M_0'(m)$ does not eliminate all infinities. Namely, there occur the so-called "b-diverg-

[†] Here and in the following G, D, Γ_5 will denote the renormalized functions and the index* will be omitted.

[††] The method of solving Eqs. (10) - (13) is essentially borrowed from reference 1, where such equations were solved for the case of quantum electrodynamics but without carrying out the renormalization of the equations themselves before solving them. In spite of the similarity of the calculation, we shall carry them out in some detail, in order to illustrate more clearly the singularities which arise in working with the renormalized equations. The results of the present work are obtained also in reference 5 which was based on the method of reference 1.

[5] A. A. Abrikosov, A. D. Galanin and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 97, 793 (1954)

ences''[6,7] whose removal is effected in perturbation theory by computing the infinite part of Γ_5, and introducing it in turn into the left and right vertices of the diagram for the mass operator. In our method of renormalization this infinity is compensated by the factor Z_1 appearing in Eq. (11). That just the function $s_1(p^2, k^2)$ is responsible for "b-divergences" can be checked, for example, in the first approximation of perturbation theory.

Following reference 1, we shall consider the vector p_μ in Eq. (10) as space-like $(p_4^2-\vec{\mathbf{p}}^2)<0$. Then displaced poles[6] will be absent and the integration on k_4 can be carried out along the imaginary axis $(k_4 \to ik_4)$. Such a transformation we shall call transition to "euclidian metric". First, we consider those parts of the mass operator which arise from the substitution of the function $\gamma_5 s_0$ for Γ_5. These will be denoted by the index $^{(0)}$. Since renormalization makes the integrals over k convergent, one can regard all integrals as carried out up to a limit of order of magnitude p. Since, furthermore, the result will only be logarithmically dependent on the upper limit, its precise value is not essential. Taking into account Eqs. (14) and (15), we get for $M_0^{(0)}(\mathbf{p})$ the following integral

$$M_0^{(0)}(\mathbf{p}) = -3\frac{g^2}{4\pi^3 i} Z_1\mathbf{p}\int\frac{1-(\mathbf{p}\mathbf{k}/p^2)}{(p-k)^2 k^2} \times F(k^2)\, s_0(k^2)\, \varphi(k^2)\, d^4k \tag{16}$$

[We exploit the slow variation of the functions F and s_0 replace their arguments $(p-k)^2$ with k^2.] The general form of the mass operator is thus:

$$M_0(\mathbf{p}) = \mathbf{p} f_1(p^2) + m f_2(p^2),$$

where f_1 and f_2 depend only on p^2. It is evident that in the renormalized expression $\mathbf{p} M_0'(m)$, the essential part for large $|p^2|$ is $\mathbf{p} f_1(m^2)$. Thus, $\mathbf{p} M_0^{(0)'}(m)$ is determined from the same Eq. (16), but under the integral sign, p^2 is set equal to m^2.

The transition to euclidian metric and the introduction of a spherical system of coordinates,

$$d^4k = 4\pi k^3 \sin^2\alpha\, dk\, d\alpha, \tag{17}$$

yields (the lower limit is of order m^2)

$$M_0^{(0)}(\mathbf{p}) = -\frac{3g^2}{2\pi^2} Z_1 \frac{\mathbf{p}}{|p^2|}\int_{m^2}^{|p^2|} dk^2 F(k^2)\, s_0(k^2)\, \varphi(k^2) \times \int_0^\pi \frac{1-(k/|p|)}{1+(k^2/|p^2|)-(2k/|p|)\cos\alpha}\sin^2\alpha\, d\alpha. \tag{18}$$

The integral over the angle α is equal to $(1/4)\pi(2-k^2/|p^2|)$, if $|k^2|<|p^2|$, and $(1/4)\pi|p^2|/k^2$ if $|k^2|>|p^2|$.

Consequently, we get

$$M_0^{(0)}(\mathbf{p}) = -\frac{3g^2}{8\pi} Z_1 \frac{\mathbf{p}}{|p^2|}\int_{m^2}^{|p^2|} dk^2 F(k^2)\, s_0(k^2)\, \varphi(k^2)\left(2-\frac{k^2}{|p^2|}\right);$$

$$\mathbf{p} M_0^{(0)'}(m) = -\frac{3g^2}{8\pi} Z_1 \mathbf{p}\int_{m^2}^{|p^2|}\frac{dk^2}{k^2} F(k^2)\, s_0(k^2)\, \varphi(k^2).$$

From these formulas, it is seen that the main term is $\mathbf{p} M_0^{(0)'}(m)$. Transforming to the logarithmic variable $z = \ln(k^2/m^2)$ and writing $\xi = \ln(-p^2/m^2)$, $\lambda = g^2/4\pi$, we have

$$M_0^{(0)}(\mathbf{p}) - M_0^{(0)}(m) - (\mathbf{p}-m) M_0^{(0)'}(m) = \frac{3}{2}\lambda Z_1 \mathbf{p}\int_0^\xi F(z)\, s_0(z)\, \varphi(z)\, dz. \tag{19}$$

We pass now to the consideration of the second part of the mass operator, which is obtained from the term Γ_5 containing s_1. We shall denote that part of $M_0(\mathbf{p})$ by the index$^{(1)}$. We get evidently

$$M_0^{(1)}(\mathbf{p}) = -\frac{3g^2}{4\pi^3 i} Z_1\gamma_5\int(\mathbf{p}-\mathbf{k})^{-1}\gamma_5\frac{\mathbf{k}\mathbf{p}}{k^2}\frac{1}{k^2} \times s_1(p^2, k^2)\, F(k^2)\, \varphi(k^2)\, d^4k, \tag{20}$$

where, as has already been proved, the essential region of tntegration lies where $|k| \gg |p|$. We can therefore neglect $\mathbf{p}$ as compared with $\mathbf{k}$ in the factor $(\mathbf{p}-\mathbf{k})^{-1}$ under the integral sign but take p as the lower limit of the integral. The result depends logarithmically on this lower limit. After a transition to euclidian metric, and integration over angle, we get

[6] F. J. Dyson, Phys. Rev. **75**, 1736 (1949)

[7] Al Salam, Phys. Rev. **82**, 217 (1951)

$$M_0^{(1)}(\mathbf{p}) = -3\lambda Z_1 \mathbf{p} \int_\xi^\infty F(z)\varphi(z)s_1(\xi, z)\,dz. \quad (21)$$

Evidently, the renormalization term is

$$(\mathbf{p}-m)\,M_0^{(1)'}(m) = \quad (22)$$

$$-3\lambda Z_1 \mathbf{p}\int_0^\infty F(z)\varphi(z)s_1(0, z)\,dz.$$

Substituting Eqs. (18), (21) and (22) into Eq. (10), we can write down a one-dimensional integral equation for the Green's function of the nucleon (after a factor $\mathbf{p}$ has been removed):

$$1+\frac{3}{2}\lambda Z_1\int_\xi^\infty F(z)\varphi(z)[s_0(z)+2s_1(\xi, z)]\,dz \quad (23)$$

$$-\frac{3}{2}\lambda Z_1\int_0^\infty F(z)\varphi(z)[s_0(z)$$

$$+2s_1(0, z)]\,dz = \frac{1}{F(\xi)}\,.$$

We turn to the construction of a one-dimensional integral equation for Γ_5, and consider, first of all, an equation for the function $s_0(p^2)$. The situation here is analogous to that occurring in the consideration of $M_0^{(0)}(\mathbf{p})$: the integral, renormalized by the subtraction of $M_1(m, \mu)$, converges for $|q|\sim|p|$ (or for $|q|$ of order of magnitude $|k|$, since $|p|$ and $|k|$ have the same order of magnitude in those Γ_5 which are necessary for the computation of the mass operator). Therefore, just as for $M_0^{(0)}(\mathbf{p})$, the main term is $M_1(m, \mu)$, which contains one more power of $\ln(p^2/m^2)$ than $M_1(\mathbf{p}, \mathbf{k})$. Considering the equation for s_0, one can neglect the small additions to Γ_5 which are proportional to s_1 since they would lead to an integral not of logarithmic form, and the result would have at worst one power of $\ln(p^2/m^2)$ less than the main term. Thus, the equation for s_0 has the form

$$s_0(p^2) = 1+\frac{g^2}{4\pi^3 i}\int(\mathbf{p}-\mathbf{q})^{-1} \quad (24)$$

$$\times\gamma_5(\mathbf{p}-\mathbf{q}-\mathbf{k})^{-1}\gamma_5 q^{-2}s_0^3(q^2)F^2(q^2)\varphi(q^2)\,d^4q\,\Big|_{\substack{p=m\\k^2=\mu^2}},$$

where the integration must be performed for $|q|\sim|p|$. Making the transition to the one-dimensional equation by the same method which was used to obtain the Green's function for the nucleon we get

$$s_0(\xi) = 1-\lambda\int_0^\xi s_0^3(z)F^2(z)\varphi(z)\,dz. \quad (25)$$

In passing, we remark that Z_1 as calculated from Eq. (7) differs from Eq. (25) only in that integration over z runs to infinity so that

$$Z_1 = s_0(\infty) = 1-\lambda\int_0^\infty s_0^3(z)F^2(z)\varphi(z)\,dz. \quad (26)$$

We consider now the equation for the function s_1. We will be interested in those momenta which are essential for the substitution of s_1 in the equation for G. We write down the integral (9) again, indicating all momenata explicitly

$$M_1(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k}) = \quad (27)$$

$$-\frac{g^2}{4\pi^3 i}\int\Gamma_5(\mathbf{p}-\mathbf{k}, \mathbf{p}-\mathbf{k}+\mathbf{q}; \mathbf{q})\,G(\mathbf{p}-\mathbf{k}+\mathbf{q})$$

$$\times\Gamma_5(\mathbf{p}-\mathbf{k}+\mathbf{q}, \mathbf{p}+\mathbf{q}; \mathbf{k})$$

$$\times\,G(\mathbf{p}+\mathbf{q})\,\Gamma_5(\mathbf{p}+\mathbf{q}, \mathbf{p}; -\mathbf{q})\,D(q^2)\,d^4q.$$

As was proved above [see Eq. (20)], the essential domain of momentum is $|k|\gg|p|$. On the other hand, the function s_1 approaches zero if the nucleon possesses momentum of that order of magnitude (below it is proved that this assertion is justified). Therefore, in the Γ_5 on the left in Eq. (27) the function s_1 need not be taken into account. In the following Γ_5, which stands in the middle, the nucleon momentum is in general of a different order of magnitude. However, if the part of Γ_5 containing the function s_1,

$$\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{k^2}\,s_1[(p+q)^2,\ k^2],$$

is written out, then one can convince oneself that the same large momenta $|q|\sim|k|$ will be essential in that integral. But then in the function $s_1[(p+q)^2, k^2]$ the arguments are of the order of magnitude unity and it approaches zero. Thus, it is necessary to keep the function s_1 only in the Γ_5 to the right. Substituting Eq. (14), and bearing in mind the equality:

$$\gamma_5(\mathbf{p}-\mathbf{k}+\mathbf{q})^{-1}\gamma_5(\mathbf{p}+\mathbf{q})^{-1}\gamma_5 =$$

$$-\gamma_5(\mathbf{p}+\mathbf{q}-\mathbf{k})^{-2}\left[1-\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{(p+q)^2}\right],$$

we get

$$M_1(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k}) \tag{28}$$
$$= \frac{g^2}{4\pi^3 i}\gamma_5 \int (\mathbf{p}+\mathbf{q}-\mathbf{k})^{-1}\left[1-\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{(p+q)^2}\right] q^{-2}$$
$$\times \left[s_0 + \frac{\mathbf{qp}}{q^2} s_1(p^2, q^2)\right] s_0 \cdot s_0 F[(p-k+q)^2]$$
$$\times\ F[(p+q)^2]\,\varphi(q^2)\, d^4q.$$

Now the terms porportional to $\mathbf{kp}$ which determine s_1 are easily separated. We get

$$\frac{\mathbf{kp}}{k^2} s_1(p^2,k^2) = \frac{g^2}{4\pi^3 i}\int \frac{d^4q}{q^2}(\mathbf{p}-\mathbf{k}+\mathbf{q})^{-2}\left[\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{(p+q)^2} s_0 + \frac{\mathbf{kp}}{(p+q)^2} s_1(p^2, q^2)\right] \times s_0 \cdot s_0 F[(p-k+q)^2] F[(p+q)^2]\,\varphi(q^2). \tag{29}$$

The integral over q converges, but, if $|k| \gg |p|$, for $|p| \ll |q| \ll |k|$ the integrand will have the form dq/q, and consequently, the integral will increase logarithmically with $|k^2|$, i.e., the function s_1 contains $\ln(k^2/p^2)$. If $|k| \sim |p|$ then the logarithmic growth of the integral does not take place and s_1 is small. This result confirms the earlier conclusion about the form of the function s_1.

In Eq. (29) the arguments of the functions s_0 are not indicated. Since the essential domain for the integration occurs for $|p| \ll |q| \ll |k|$ the argument of the first function s_0 is q^2, but the arguments of the other two functions s_0 are equal to k^2. The arguments of the functions F can be taken: one equal to k^2, the other equal to q^2. Because the integral determining the function s_1 converges, it is not necessary to take into account the renormalization term $M_1(m, \mu)$.

We pass to euclidian metric and carry out the integral over angle.. Here p^2 and q^2 can be neglected as compared with k^2, and p^2 as compared with q^2 (but not $\mathbf{q}$ with respect to $\mathbf{p}$ in the term $\mathbf{p}+\mathbf{q}$). The limit of the q^2 integration can be taken as p^2 or k^2, which is not essential. We get the result

$$s_1(\xi, \eta) = \frac{1}{2}\lambda s_0^2(\eta) F(\eta)\int_\xi^\eta [s_0(z) + 2s_1(\xi, z)]\, F(z)\,\varphi(z)\, dz, \tag{30}$$

where

$$\xi = \ln\left(-\frac{p^2}{m^2}\right);\quad \eta = \ln\left(-\frac{k^2}{m^2}\right).$$

Equation (30) can be solved to yield $s_1(\xi, \eta)$ in terms of s_0, F and ϕ. To do this conveniently[1], we introduce the function

$$q(\xi, \eta) = \frac{s_1(\xi, \eta)}{s_0^2(\eta) F(\eta)}. \tag{31}$$

From Eqs. (30) and (25), we find easily

$$\frac{\partial}{\partial\eta}[s_0(\eta)\, q(\xi, \eta)] = \frac{1}{2}\lambda s_0^2(\eta) F(\eta)\,\varphi(\eta), \tag{32}$$

whence

$$s_0(\eta)\, q(\xi, \eta) = \frac{1}{2}\lambda\int_\xi^\eta s_0^2(z) F(z)\,\varphi(z)\, dz; \tag{33}$$

from which we learn that $q(\xi, \xi) = 0$. Thus,

$$s_1(\xi, \eta) = \frac{1}{2}\lambda s_0(\eta) F(\eta)\int_\xi^\eta s_0^2(z) F(z)\,\varphi(z)\, dz. \tag{34}$$

Now with the help of Eq. (34), we can prove the correctness of our previous assertion that the substitution of s_1 in the equation for the nucleon Green's function leads to an integral which diverges even after subtraction of the terms $M_0(m)$ and $(\mathbf{p}-m) M_0'(m)$. In fact, we differentiate Eq. (23) with respect to ξ. Then, if the subtraction of $M_0(m)$ and $(\mathbf{p}-m) M_0'(m)$ were to make the integral converge, after one (or at most, two) differentiations, we should get a finite expression. However, as is not difficult to see from Eqs. (23) and (34), after an arbitrary number of differentiations, the integral on ξ in Eq. (23) will give rise to logarithmic divergences. That confirms our assertion that momenta satisfying $|k^2| \gg |p^2|$ play a fundamental role in that part of the mass operator which contains $s_1(p^2, k^2)$.

3. EQUATIONS FOR THE MESON GREEN'S FUNCTION

We consider first of all the small correction to Γ_5, which must be taken into account in solving Eqs. (12) and (13) for the Green's function of the meson. Because the polarization operator diverges quadratically, small corrections to Γ_5, decreasing as $1/p^2$, still lead to logarithmically diverging integrals and must be taken into account. On the other hand, the function s_1 can be neglected in the polarization operator. In fact, in Eq. (13)

there appears $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}, \mathbf{k})$ in which the additions will be small for the case which interests us, $|p^2| \gg |k^2|$. Thus, in complete analogy with the case of the mass operator, the substitution of the small corrections to Γ_5 into the polarization operator leads to an integral which diverges after the subtraction of $P_0(\mu^2)$ and $(k^2-\mu^2)P_0'(\mu^2)$ ["b-divergences" in the polarization operator which are compensated by the factor Z_1 in Eq. (13)]. Therefore, it is not necessary to take into account the function $s_1(p^2, k^2)$ which approaches zero for equal nucleon momenta.

Consequently, it is necessary to substitue $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}, \mathbf{k})$ into Eq. (13) in the following form:

$$\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) = \gamma_5\left[s_0 + \frac{k^2}{p^2+k^2}\, s_2(k^2, p^2)\right], \tag{35}$$

where $s_2(k^2, p^2)$ is a slowly varying function of its arguments. It can be shown that it is not necessary to take into account small corrections of the form $[\mathbf{kp}(kp)/p^4]\, s_3$. In fact, considering the inhomogeneous term in the equation for s_3, i.e., the expression obtained by replacing Γ_5 by $\gamma_5 s_0$ on the right hand side of Eq. (9), one can convince oneself that in the factor of $\mathbf{kp}(kp)/p^4$ there does not occur $\ln(p^2/k^2)$ for $|p^2| \gg |k^2|$. This means that it is not necessary to take into account additional terms in Γ_5 of such a form.

We construct the equation for $s_2(k^2, p^2)$ for $|p^2| \gg |k^2|$. We write $M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})$ indicating all momenta explicitly.

$$M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) = -\frac{g^2}{4\pi^3 i}\int \Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{q}; -\mathbf{q})\, G(\mathbf{p}-\mathbf{q}) \times \Gamma_5(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}+\mathbf{k}; \mathbf{k})\, G(\mathbf{p}-\mathbf{q}+\mathbf{k}) \times \Gamma_5(\mathbf{p}-\mathbf{q}+\mathbf{k}, \mathbf{p}+\mathbf{k}; \mathbf{q})\, D(q^2)\, d^4q. \tag{36}$$

The function $s_2(k^2, p^2)$ is equal to zero (to logarithmic approximation) if the momentum of the meson k^2 is of the same order of magnitude as the momentum of the nucleon p^2 (this is suggested by perturbation theory and confirmed by the following results). Thus, it follows that in Eq. (36) the function s_2 need appear only in the middle one of the three Γ_5, and in the two outside ones, one can set $\Gamma_5 = \gamma_5 s_0$. We make the change of variable $\mathbf{p}-\mathbf{q} = \mathbf{q}'$ in Eq. (36). The rapidly varying factor in the functions G, D will equal

$$(\mathbf{p}-\mathbf{q})^{-1}(\mathbf{p}-\mathbf{q}+\mathbf{k})^{-1}q^{-2} = \left[1 + \frac{\mathbf{q}'\mathbf{k}}{q'^2}\right]\frac{1}{(q'+k)^2(p+q')^2}. \tag{37}$$

Bearing in mind that $|k^2| \ll |q'^2| \ll |p^2|$, it is easy to figure out the arguments of all slowly varying functions:

$$M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) = \frac{g^2}{4\pi^3 i}\gamma_5\int\left[1+\frac{\mathbf{q}'\mathbf{k}}{q'^2}\right]\frac{d^4q'}{(q'+k)^2(p+q')^2} \times s_0^2(p^2)\, F^2(q'^2)\varphi(p^2)\left[s_0(q'^2) + \frac{k^2}{q'^2}\, s_2(k^2, q'^2)\right]. \tag{38}$$

In the inhomogeneous terms of the equation for s_2 [i.e., those containing $s_0(q'^2)$] only the second term in the square bracket of Eq. (37) can give a contribution proportional to $(k^2/p^2)\ln(p^2/k^2)$. On the other hand, in terms proportional to s_2 in Eq. (38), one need only keep the first term in the square brackets in Eq. (37). Thus, we get the following equation for the function s_2:

$$\frac{k^2}{p^2}\, s_2(k^2, p^2) = \frac{g^2}{4\pi^3 i}\int\frac{d^4q'}{(q'+k)^2(p+q')^2}\, s_0^2(p^2)\, F^2(q'^2)\,\varphi(p^2) \times\left[\frac{\mathbf{q}'\mathbf{k}}{q'^2}\, s_0(q'^2) + \frac{k^2}{q'^2}\, s_2(k^2, q'^2)\right], \tag{39}$$

where the integration should be from $|q'| \sim |k|$ to $|q'| \sim |p|$. Making the transition to euclidian metric and integrating over angle, we get the following one-dimensional integral equation for s_2:

$$s_2(\xi, \eta) = -\frac{\lambda}{2}\, s_0^2(\eta)\,\varphi(\eta)\int_\xi^\eta [s_0(z) - 2s_2(\xi, z)]\, F^2(z)\, dz, \tag{40}$$

where $\xi = \ln(-k^2/m^2)$; $\eta = \ln(-p^2/m^2)$. The solution of Eq. (40) is carried out analogously to the solution of Eq. (30) for s_1. We set

$$p(\xi, \eta) = \frac{s_2(\xi, \eta)}{s_0^2(\eta)\,\varphi(\eta)}. \tag{41}$$

Then

$$\frac{\partial}{\partial\eta}[s_0(\eta)\, p(\xi, \eta)] = -\frac{\lambda}{2}\, s_0^2(\eta)\, F^2(\eta) \tag{42}$$

and

$$s_0(\eta)\, p(\xi, \eta) = -\frac{\lambda}{2}\int_{\xi}^{\eta} s_0^2(z)\, F^2(z)\, dz. \quad (43)$$

Thus

$$s_2(\xi, \eta) = -\frac{\lambda}{2} s_0(\eta)\, \varphi(\eta)\int_{\xi}^{\eta} s_0^2(z)\, F^2(z)\, dz. \quad (44)$$

From Eq. (44) it is seen that $s_2(k^2, p^2)$ approaches zero for $|k^2| \sim |p^2|$. In addition, it is easily verified, that after the substitution of s_2 in the polarization operator, an infinite expression is obtained even after the subtraction of the quantities $P_0(\mu^2)$ and $(k^2 - \mu^2)P_0'(\mu^2)$.

Finally, we reduce the equation for the Green's function of the meson to one-dimensional form. Here there is a complete analogy with the structure of the one-dimensional equation for the Green's function of the nucleon.

If $\gamma_5 s_0$ is taken in place of $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})$, then the resulting integral will converge for $|p^2| \sim |k^2|$ thanks to the renormalization [subtraction of the quantities $P_0(\mu^2)$ and $(k^2-\mu^2) \times P_0'(\mu^2)$]. Here the terms $P_0(k^2)$ and $P_0(\mu^2)$ can be neglected on the same grounds as in the equation for $G(\mathbf{p})$. The remaining term, $(k^2-\mu^2) \times P_0'(\mu^2)$ reduces to a one-dimensional integral by repeated use of our method and gives

$$-(k^2-\mu^2)\, P_0^{(0)'}(\mu^2) = 4\lambda Z_1 k^2 \int_0^{\xi} s_0(z)\, F^2(z)\, dz. \quad (45)$$

In the integral, the quantity $\gamma_5(k^2/p^2)s_2(k^2,p^2)$ which replaces $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})$ plays a role for momenta $|p^2| \gg |k^2|$. Proceeding in precisely the same manner as in the derivation of the equation for $G(\mathbf{p})$, we get

$$P_0^{(1)}(k^2) - (k^2-\mu^2)\, P_0^{(1)'}(\mu^2) = -8\lambda Z_1 k^2 \int_{\xi}^{\infty} F^2(z)\, s_2(\xi, z)\, dz + 8\lambda Z_1 k^2 \int_0^{\infty} F^2(z)\, s_2(0, z)\, dz. \quad (46)$$

Gathering together Eqs. (12), (13), (45) and (46), we get the one-dimensional integral equation for the Green's function of the meson

$$1 + 4\lambda Z_1 \int_{\xi}^{\infty} F^2(z)\,[s_0(z) - 2s_2(\xi, z)]\, dz - 4\lambda Z_1 \int_0^{\infty} F^2(z)\,[s_0(z) - 2s_2(0, z)]\, dz = \frac{1}{\varphi(\xi)}. \quad (47)$$

4. THE ASYMPTOTIC GREEN'S FUNCTION OF NUCLEON AND MESON

We solve the system of Eqs. (23), (25), (30), (40) and (47). With the help of Eqs. (30) and (31), we write Eq. (23) thus:

$$\frac{1}{F(\xi)} = 1 + 3Z_1 q(\xi, \infty) - 3Z_1 q(0, \infty), \quad (48)$$

and, with the help of Eqs. (40) and (41), Eq. (47) thus:

$$\frac{1}{\varphi(\xi)} = 1 - 8Z_1 p(\xi, \infty) + 8Z_1 p(0, \infty) \quad (49)$$

Differentiating Eqs. (48) and (49) with respect to ξ and using Eqs. (33), (43) and (26), we find

$$\frac{d}{d\xi}\frac{1}{F(\xi)} = 3Z_1 \frac{\partial q(\xi, \infty)}{\partial \xi} = -\frac{3}{2}\lambda s_0^2(\xi)\, F(\xi)\, \varphi(\xi), \quad (50)$$

$$\frac{d}{d\xi}\frac{1}{\varphi(\xi)} = -8Z_1 \frac{\partial p(\xi, \infty)}{\partial \xi} = -4\lambda s_0^2(\xi)\, F^2(\xi), \quad (51)$$

while differentiation of Eq. (25) gives

$$\frac{ds_0(\xi)}{d\xi} = -\lambda s_0^3(\xi)\, F^2(\xi)\, \varphi(\xi). \quad (52)$$

The initial conditions for the differential Eqs. (50) - (52) follow from the integral Eqs. (23), (25) and (47):

$$F(0) = s_0(0) = \varphi(0) = 1. \quad (53)$$

Equations (50) - (52) are easily solved. The solutions corresponding to the initial conditions (53), have the form:

$$F(\xi) = (1 - 5\lambda\xi)^{-3/10}, \quad (54)$$
$$s_0(\xi) = (1 - 5\lambda\xi)^{1/5}$$
$$\varphi(\xi) = (1 - 5\lambda\xi)^{-4/5}:$$

The substitution of these expressions (54) in the integral equations shows thay they are indeed solutions, provided that the path of integration around the singularities is chosen consistent with

$$\xi = \ln\left(-\frac{p^2}{m^2(1-i\varepsilon)}\right) = \ln\left(-\frac{p^2}{m^2}\right) + i\varepsilon,$$

where $\epsilon > 0$.

For small $\lambda\xi$, one can expand Eq. (54) in series in $\lambda\xi$. The result of a calculation of G, Γ_5 and D by perturbation theory, coincides with the series for Eq. (54) (this was verifed up to terms of second order inclusively). We mention that in the solution of the integral equation the factor Z_1 cancels, and the solution (54) appears finite and free from ambiguities of any kind.

The solution (54) for the Green's function of nucleon and meson has a pole * at the point $\ln(-p^2/m^2) = 1/(5\lambda)$. Physically, the existence of such a pole indicates the appearance of a meaningless solution, because the mass κ, corresponding to such a pole is imaginary [in order that $p^2 = \kappa^2$ may satisfy the equation $\ln(-\kappa^2/m^2) = 1/(5\lambda)$]. The meaninglessness of the solution near or at the pole can be seen from another point of view. The experimental charge g is connected with the fictive charge g_0 by the correspondence $g_0^2 = Z_1^2 Z_2^{-2} Z_3^{-1} g^2$. Clearly, both the experimental charge g and the fictive charge g_0 should be real quantities (the conclusion follows from the hermitean character of the interaction hamiltonian). Consequently, the product $Z_1 Z_2^{-1} Z_3^{-½}$ is necessarily real and positive. But, calculating it with the aid of the solution (54), and Eqs. (5) and (55), one can convince oneself that $Z_1 Z_2^{-1} Z_3^{-½}$ is complex if Eq. (54) is taken sufficiently close to its pole.

Thus, we arrive at a contradiction whose resolution is that the solution (54) is valid only up to the pole. In fact, considering the cross-over diagrams we have neglected, for example, the one depicted in Fig. 3, one can see that the ratio of the contribution from them and the contribution from the diagrams we have taken into account will be of order of magnitude

$$\frac{\lambda}{(1-5\lambda\xi)^{\alpha}},$$

where α is some positive fractional number. Therefore, close to the pole where $1 - 5\lambda\xi \lesssim \lambda$, the contribution from the cross-over diagram will not be small, and our solutions lose their meaning. Physically, this corresponds to the fact that at small distances, interaction with a small coupling constant (which would give weak interaction at ordinary distances) becomes strong. Thus, for the construction of the asymptotic Green's functions for such large momenta it is necessary to take into account a significantly larger number of diagrams than we have done when we omitted all cross-over diagrams.

A more precise determination of the limits of applicability at high momenta of the formula we have obtained requires additional analysis, since, for $1 - 5\lambda\xi \sim \lambda$ in the expansion of Eq. (54) in powers $(\lambda\xi)^n$, terms with $n \sim \lambda^{-1}$ turn out to be essential. In such a situation the asymptotic series of perturbation theory already begin to diverge. The circumstance that the method of renormalization considered in reference 2 led to the removal of infinities in the above problem permits one to hope that it may be applicable in the solution of more complicated problems.

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* In neutral pseudoscalar theory, the solution has the following form:

$$F(\xi) = (1-5\lambda\xi)^{-1/10};$$
$$s_0(\xi) = (1-5\lambda\xi)^{-1/5};$$
$$\varphi(\xi) = (1-5\lambda\xi)^{-2/5}.$$

A pole also appears in this variant of the theory. It seems that the presence of this pole is due to the same general cause as in symmetrical theory.

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