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# The Conditions for Self-Excitation of a Singing Flame

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The problem of the stability of a singing flame is considered, beginning with the considerations of Rayleigh, and taking into account the phenomenological delay in burning.

THE phenomenon of the singing flame was observed as early as the second half of the eighteenth century and since that time it has frequently served as the subject of an effective lecture demonstration. If a gas burner is placed inside a tube, as is shown in Fig. 1, then, under a number of conditions, intense vibrations of the flame and of the surrounding air are set up, and the tube begins to resound, or, as is said, the flame begins to sing.

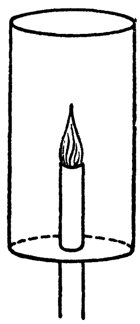

FIG. 1

This phenomenon has been observed by a number of authors [1] . Different simple hypotheses as to the reasons for the onset of vibrations have been given. The first correct, qualitative explanation of the mechanism for maintaining the vibrations was given by Rayleigh[2], who showed how the vibrating flame could maintain sound vibrations in the tube. The flame maintains the vibrations of the air column if it is located near an anti-node of pressure and vibrates so that, at the moment of compression, a larger amount of heat is evolved than at the moment of rarefaction. In subsequent researches, this qualitative picture has not been changed in any essential way, in spite of the large amount of research on the problem*.

In the present work, which is based on the considerations of Rayleigh[2], and which takes into account the phenomenological delay in burning, a detailed study is carried out on the conditions for the self-excitation of the singing flame. The results which are obtained are in excellent qualitative agreement with the known experimental facts, but they do not provide a quantitative check.

## 1. DISCRETE MODEL OF A SINGING FLAME (MODEL No. 1)

A study of the conditions for self-excitation of a singing flame requires the consideration of small vibrations which are superimposed on the established gas flow in the system. In the discrete model, if we idealize the sounding (air) and supplying (gas) tubes in the form of resonators, we obtain the arrangement shown in Fig. 2. Each of these resonators, for vibration frequencies $\omega \ll a/l$, where $l$ is a linear dimension of the resonator, $a$ the sound velocity, is itself an oscillator, the mass in which is the mass of the air vibrating in the neck of the resonator, and the elasticity is that of the air trapped within the resonator and compressed by the air, vibrating like a piston in the neck.

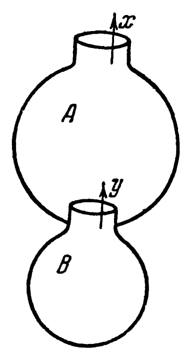


FIG. 2

Let $x$ be a small displacement of the air in the neck of the resonator $A$ (for graphic purposes, this displacement can be represented as a change in position of a piston in the neck of the resonator), and let $y$ be the displacement of the gas in the

* This work was complete in 1952 (see the report of GIFTI for 1952). In 1953 B. V. Raushenbakh[3] considered the problem of the excitation of vibrations in the case of slow propagation down the tube.

[1] A. T. Jones, J. Acoust. Soc. Am. 16, 254 (1945)

[2] Rayleigh, *Theory of Sound*, v. II, pp 222-228

[3] B. V. Raushenbakh, Zh. Tekhn. Fiz. 23, 358 (1953)

neck of resonator $B$. Also, let $m_1$ and $m_2$ be the "masses" of vibrating gas in the necks of resonators $A$ and $B$, and $p$, $p_1$ and $p_2$ be the external pressure, and the pressures in resonators $A$ and $B$, respectively. Then the equation of motion of the masses of air in the necks can be written, for small vibrations (for those characteristic frequencies which satisfy the boundary conditions given above), as

$$m_1\ddot{x} + \delta_1\dot{x} = S_1\Delta p_1; \qquad (1.1)$$
$$m_1\ddot{y} + \delta_2\dot{y} = S_2(\Delta p_2 - \Delta p_1),$$

where $S_1$ and $S_2$ are the cross sectional areas of the resonator necks. In the absence of flame, a mutual elastic coupling takes place between the two oscillators $x$ and $y$. Actually, a change in $x$ produces a change in $p_1$ proportional to $S_1x$, and a change in $y$ produces changes in $p_1$ and $p_2$ proportional to $S_2y$, i.e.,

$$\Delta p_1 = -k_1S_1x + k_1S_2y, \quad \Delta p_2 = -k_2S_2y,$$

which, upon substitution into Eq. (1.1), yields

$$m_1\ddot{x} + \delta_1\dot{x} = -k_1S_1^2x + k_1S_1S_2y; \qquad (1.2)$$
$$m_2\ddot{y} + \delta_2\dot{y} = -(k_1 + k_2)S_2^2y + k_1S_1S_2x.$$

The second terms in the right hand sides of Eqs. (1.2) indicate the elastic coupling between the oscillators $x$ and $y$. It is easy to prove that small vibrations, described by the differential equations of Eq. (1.2), die out with the passage of time, i.e., the initial system is stable.

With the appearance of the flame, the character of the coupling between the oscillators is considerably changed, and in this change lies the reason for the possible self-excitation of vibrations. Jumping ahead somewhat, we can say that the system under study consists of two dissipative parts— the sounding tube and the tube supplying the gas — between which there exists nonconservative coupling which transfers its energy into the singing flame. The problem of the investigation of the excitation condition of the singing flame is primarily the problem of the investigation of this coupling.

We assume that the rate of heat production of the flame (i.e., the rate of the chemical reaction of combustion) is proportional to the rate of flow of the gas, and takes place with a certain phenomenological delay of burning, $\tau$ ( if a liquid fuel had been used, then $\tau$ would have been a quantity of the order of the time during which burning of the droplet takes place after its emission from the jet). We shall also assume that the heat loss in the burning, due to conduction and convection to the outside, remains constant. Then, in the presence of the flame, just as earlier, a change in $y$ produces a large change in the pressure $p_1$, since the gas emerging from the tube, in burning, produces a positive pressure amplification.

In accordance with our assumptions, we now have

$$\Delta p_1 = -k_1S_1x + k_1S_2y + \alpha k_1S_2y_\tau$$

($\alpha$ is a constant of proportionality, $y_\tau$ is the value of $y$ at the time $t - \tau$), which, after substitution in Eq. (1.1), yields

$$m_1\ddot{x} + \delta_1\dot{x} = -k_1S_1^2x + k_1S_1S_2y + \alpha k_1S_1S_2y_\tau; \qquad (1.3)$$
$$m_2\ddot{y} + \delta_2\dot{y} = -(k_1 + k_2)S_2^2y + k_1S_1S_2x.$$

If we denote the characteristic frequencies of the resonators $A$ and $B$ by $\omega_1$ and $\omega_2$, and neglect the term $k_1S_1S_2y$ in comparison with $\alpha k_1S_1S_2y_\tau$, then we get the following linearized equations for small vibrations:

$$\ddot{x} + h_1\dot{x} + \omega_1^2x = \mu y_\tau; \qquad (1.4)$$
$$\ddot{y} + h_2\dot{y} + \omega_2^2y = \nu x.$$

Here

$$h_1 \equiv \frac{\delta_1}{m_1}, \quad \omega_1^2 \equiv \frac{k_1S_1^2}{m_1}, \quad \mu \equiv \frac{\alpha k_1S_1S_2}{m_1},$$

$$h_2 \equiv \frac{\delta_2}{m_2}, \quad \omega_2^2 \equiv \frac{(k_1 + k_2)S_2^2}{m_2}, \quad \nu = \frac{k_1S_1S_2}{m_2}.$$

The characteristic equation of this system, obtained by the usual methods, is

$$(z^2 + h_1z + \omega_1^2)(z^2 + h_2z + \omega_2^2) - \nu\mu e^{-\tau z} = 0. \qquad (1.5)$$

Solution for the parameter $\mu\nu$ is easily accomplished. Setting $z = i\omega$ in Eq. (1.5), we get

$$\nu\mu = (\omega_1^2 - \omega^2 + ih_1\omega)(\omega_2^2 - \omega^2 + ih_2\omega)e^{i\tau\omega} \qquad (1.6)$$

For $\tau = 0$

$$(\nu\mu)_0 = (\omega_1^2 - \omega^2 + ih_1\omega)(\omega_2^2 - \omega^2 + ih_2\omega).$$

The curve $(\nu\mu)_0$ is plotted in Fig. 3*a*. The abscissa of point $b$, the intersection of the curve with the real axis, is given by

$$-\frac{h_1h_2}{h_1 + h_2}\left[(\omega_1^2 - \omega_2^2) + h_1\omega_2^2 + h_2\omega_1^2\right],$$

and decreases in value as $\omega_1$ approaches $\omega_2$.

With a delay $\tau$, the curve of Fig. 3*a* becomes "twisted" and takes on the form shown in Fig. 3*b*. The region of stability is denoted by the thick line. The system will be stable for an arbitrary delay if $|\nu\mu| < r_{min}$, where $r_{min}$ is the minimum distance from the points of the curve of Fig. 3*a* to the

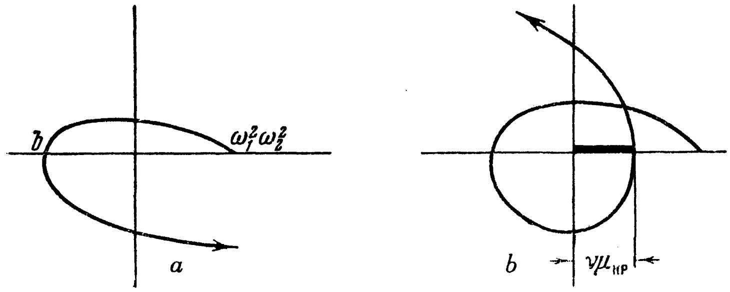


FIG. 3. $a - (\nu\mu)_0$, $b - (\nu\mu)$

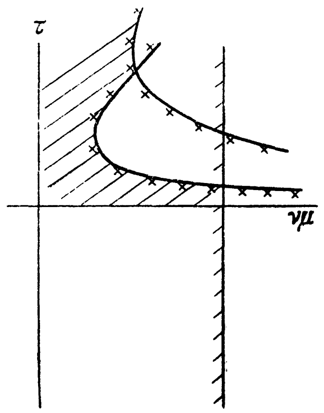


FIG. 4

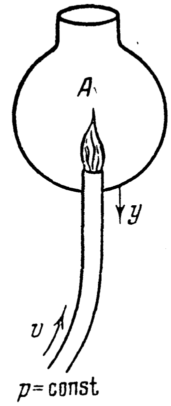


FIG. 5

origin. The plot in the plane of the parameters $\nu\mu$ and $\tau$ is represented in Fig. 4, where the region of stability is indicated by shading[4].

It follows from Figs. 3*a*, 3*b* and 4 that (a) there is no instability for small values of $\nu\mu$ without delay $\tau$; (b) excitation of the vibrations is due to increase in $\nu\mu$, to the closeness of the frequencies $\omega_1$ and $\omega_2$, and also to a decrease in the damping coefficients $h_1$ and $h_2$.

## 2. SEMIDISTRIBUTED MODEL OF THE PHENOMENON (MODEL No. 2)

Gas is fed into the resonator $A$ by a long narrow tube $B$ (with cross sectional area $\sigma$) from the resonator $C$ (Fig. 5). It is required to find the conditions for self-excitation of vibrations of the flame, assuming that, just as before, the resonator $A$ is a discrete section, but account is now taken of wave phenomena in the delivery tube $B$.

The equations for small vibrations of the gas in the neck of the resonator have the form

$$m\ddot{x} + \delta\dot{x} = S\Delta p, \tag{2.1}$$

and the equations for small vibrations of the gas which are superimposed on the flow existing in the supply tube are now described by equations with partial derivatives:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\partial v}{\partial y} = -\frac{1}{\rho a^2}\frac{\partial p}{\partial t}, \tag{2.2}$$

where $v$ is the change in the velocity of the gas in the tube, $p$ is the change in pressure, $\rho$ is the gas density, and $a$ is the velocity of sound in the gas.

To Eq. (2.2) we add the following boundary conditions at $y = L$ and $y = 0$: we assume the pressure to be constant in the reservoir $C$, or

$$p_{y=L} = 0; \tag{2.2'}$$

For $y = 0$, an efflux condition exists which, after linearization, can be put in the form

$$v = \nu(p - \Delta p), \tag{2.2''}$$

where $\nu = \alpha_1^2 v_0/\rho$ ($\alpha_1$ = coefficient of delivery, $v_0$ = velocity of steady gas flow).

As in the previous model,

$$\Delta p = \alpha\int v_\tau dt + \beta\int v dt - \gamma x, \tag{2.3}$$

where $v$ is the change in the discharge velocity of

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[4] Iu. I. Neimark, *Stability of Linear Systems*, Moscow, (1949)

the gas in the supply tube from its stationary value at the time $t$, and $v_\tau$ is the corresponding value at the time $t-\tau$; $\alpha$, $\beta$ and $\gamma$ are constants, equal, respectively, to

$$\alpha = \frac{\rho\sigma nRq}{c_v M_1 V}, \quad \beta = \frac{\sigma p_0}{V}, \quad \gamma = \frac{Sp_0}{V}.$$

Here $M_1$ is the mass of the mixture of gaseous combustion products and air in the resonator, $q$ the heat capacity of the gas, $n$ the number of gram molecules, $R$ the gas constant, $\rho$ the density of the gas, $p_0$ the stationary pressure in the resonator, and $V$ the volume of the resonator. Actually, by virtue of the linearization, the overall change in pressure $\Delta p$ in the resonator is composed of changes produced by

a) the influx of gas through the tube $B$, equal to $\frac{\sigma p_0}{V}\int v\,dt$;

b) the discharge through the neck of the resonator $A$, equal to

$$\frac{Sp_0}{V}\int \dot{x}\,dt = \frac{Sp_0}{V}x;$$

c) the generation of excess heat from the burning gas, equal to**

$$\frac{\sigma\rho nRg}{c_v M_1 V}\int v_\tau dt;$$

d) a change $\Delta p_{\text{chem}}$ as a result of the chemical reaction. We can neglect this latter term since, for hydrogen,

$$\left|\frac{\Delta p_{\text{chem}}}{\Delta p_T}\right| \approx \frac{1}{80}.$$

Therefore the complete system of linearized equations for the semidistributed model will be

$$m\ddot{x} + \delta\dot{x} = S\Delta p_{y=0} \tag{2.4}$$

$$= S\left[\alpha\int v_\tau dt + \beta\int v\,dt - \gamma x\right]_{y=0};$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\partial v}{\partial y} = -\frac{1}{\rho a^2}\frac{\partial p}{\partial t};$$

$$p_{y=L} = 0, \quad v_{y=0} = \nu(p - \Delta p).$$

In order to find the characteristic equation of the system of equations (2.4), we follow reference 5 and write

$$p(y,t) = (Ae^{-zy/a} + Be^{zy/a})e^{zt},$$
$$v(y,t) = \frac{1}{a\rho}[-Ae^{-zy/a} + Be^{zy/a}]e^{zt},$$
$$x = Ce^{zt},$$

which, after substitution into the first, third and fourth members of Eq. (2.4) reduces to the system of equations

$$C(mz^2 + \delta z) = S\Big\{[\alpha(-A+B)e^{-z\tau} + \beta(-A+B)]\frac{1}{a\rho z} - \gamma C\Big\},$$

$$Ae^{-zL/a} + Be^{zL/a} = 0,$$

$$\frac{1}{a\rho}(-A+B) = \nu\Big\{A + B - \left[\frac{\alpha}{a\rho z}e^{-z\tau}(-A+B) + \frac{\beta}{a\rho z}(-A+B) - \gamma C\right]\Big\}.$$

We eliminate the constants $A$, $B$ and $C$ from this system, and obtain the characteristic equation

$$(z^2 + hz + k^2)\left(1 + D\,\text{th}\frac{z\tau_1}{2}\right) + (z+h)E(1 + \mu e^{-\tau z}) = 0. \tag{2.5}$$

Here

$$h = \frac{\delta}{m}; \quad k^2 = \frac{\gamma S}{m}; \quad D = \nu a\rho;$$

$$E = \nu\beta; \quad \mu = \frac{\alpha}{\beta}; \quad \tau_1 = \frac{2L}{a}.$$

The system under consideration will be stable if all the roots of the characteristic equation (2.5) lie to the left of the imaginary axis. On the plane of the parameters $D$ and $\tau$, we find regions where this condition is satisfied, and regions where it is not. From Eq. (2.5) it follows that

$$D = -\left[\frac{P_1}{P_2}(1 + \mu e^{-\tau z}) + 1\right]\text{cth}\frac{z\tau_1}{2}, \tag{2.6}$$

$$P_1 \equiv E(z+h), \quad P_2 \equiv z^2 + hz + k^2.$$

In Eq. (2.6) we set $z = i\omega$. Then (2.7)

$$D = i\left[\frac{P_1(i\omega)}{P_2(i\omega)}(1 + \mu e^{-i\tau\omega}) + 1\right]\text{ctg}\frac{\omega\tau_1}{2} \equiv Q\,\text{ctg}\frac{\omega\tau_1}{2}.$$

The boundary of the region of instability consists of points which satisfy Eq. (2.7), at least for one

** In burning for a time $t$, the mass $M = \sigma\rho\int v_\tau dt$ of combustible gas which reacts with the mass $\epsilon M$ of air ($\epsilon$ is a coefficient defined by the chemical equation of the combustion process). As a result of the burning, we obtain $(1+\epsilon)M$ products of combustion plus the heat $qM$. Because of the rise in temperature, the pressure in the resonator will be increased by an amount given by:

$$\Delta p_T = \frac{nR}{V}\Delta T = \frac{nR}{V}\frac{qM}{c_v M_1} = \frac{\sigma\rho nRq\int v_\tau dt}{Vc_v M_1}$$

5 Iu. I. Neimark, Uch. Zap. Gorki State Univ. 14, 191 (1950)

real $\omega$. Such $\omega$ must either be a root of the equation

$$\operatorname{ctg}(\omega\tau_1/2) = 0,$$

or a value which converts the expression

$$Q \equiv i\left[\frac{P_1(i\omega)}{P_2(i\omega)}(1+\mu e^{-i\tau\omega})+1\right] \tag{2.8}$$

into a purely real number (because $D$ is real). For each of the roots of the equation, cotan $(\omega\tau_1/2) = 0$, i.e.,

$$\omega = (2n+1)\pi/\tau_1 \tag{2.9}$$

so that $D = 0$.

We construct the curve $Q(i\omega)$ to find the value of $\omega$ for which $Q$ is a real number. For $\tau = 0$, the shape of this curve is shown in Fig. 6 *a* (only the part of this curve corresponding to $\omega > 0$ is shown; the second half of the curve, for $\omega < 0$, is located symmetrically with respect to the imaginary axis). Its projection, which increases with increase in the damping $h$, corresponds approximately to the characteristic frequency of the resonator for small $h$. The projection also increases for an increase in $\mu$, since $\mu \gg 1$, with the center of symmetry at the point $(0, i)$. The presence of $\tau$ produces a rotation of the points of this curve through an angle $-\omega\tau$ about the point $(0, i)$. As a result, the points on it can intersect with the real axis (Fig. 6 *b*).

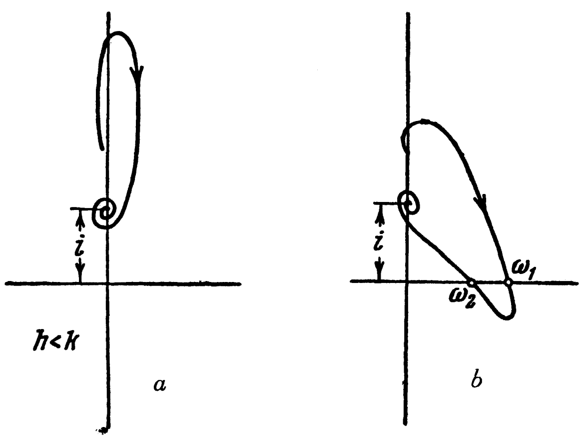


FIG. 6. $a - (Q)_0$, $b - (Q)$

The equation for the determination of the corresponding points of intersection $\omega = \omega_j$ is obtained by setting the imaginary part of $Q(i\omega)$ equal to zero. The condition $\operatorname{Im}Q(i\omega) = 0$ gives

$$(k^2-\omega^2)^2 + h^2\omega^2 + E\,[(1+\mu\cos\omega\tau)\,hk^2 + \mu\,\omega\sin\omega\tau\,(k^2-\omega^2-h^2)] = 0. \tag{2.10}$$

After the $\omega_j$ are established, the equation of the boundary of the stable region in the plane of $D, \tau_1$ is written in the form

$$D = Q(\omega_j)\operatorname{ctg}(\omega_j\tau_1/2).$$

After drawing in the shaded lines, in accordance with the rules set forth in reference 4, we obtain the plot shown in Fig. 7. The region of stability is shaded on them. The form of the auxiliary curve is obtained from these drawings:

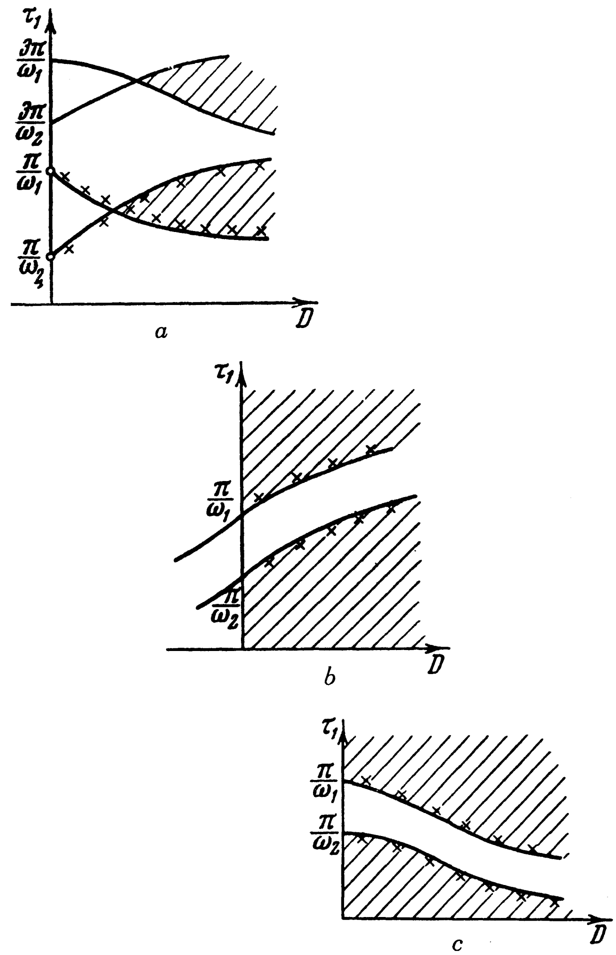


FIG. 7.

a) self-excitation is not possible without delay in burning;

b) an alternation of stability and instability takes place initially upon an increase in the length of the supply tube, but for a sufficiently long tube, instability is approached (we note that instability is possible even in the absence of the supply tube, i.e., for $\tau_1 = 0$);

c) for small damping and not very large $\mu$, the frequency of the vibration that is excited is close to the resonant frequency of vibration of the resonator;

d) a decrease in $h$ and an increase in $E$ or $\mu$ favor the establishment of instability.

## 3. DISTRIBUTED MODEL OF THE SINGING FLAME WITHOUT CONSIDERATION OF CONVECTION (MODEL No. 3)*

We shall now consider the supply tube and the sounding tube as one-dimensional distributed links, and the flame as a distributed, linear heat source. We break up the system under consideration into four parts: the burning zone 2, the zones 1 and 3, in which we neglect the changes in temperature brought about by the vibrations of the flame, and the supply tube 4 (Fig. 8).

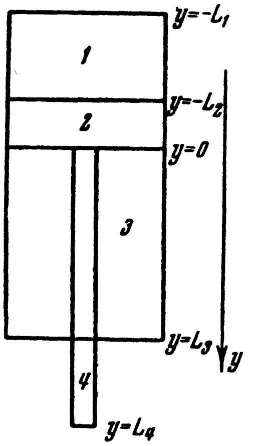


FIG. 8

The changes in velocity and pressure in regions 1,3 and 4 (if terms of the order of the ratio of the velocity of convective flow to the velocity of sound are neglected) are described by the linearized equations

$$\frac{\partial v_i}{\partial t} = \frac{1}{\rho_{i0}}\frac{\partial p_i}{\partial y}, \quad \frac{\partial p_i}{\partial t} = \rho_{i0} a_{i0}^2 \frac{\partial v_i}{\partial y} \quad (i = 1, 3, 4). \tag{3.1}$$

In the burning zone, we have the linearized Euler equation

$$\frac{\partial v_2}{\partial t} = \frac{1}{\rho_{20}}\frac{\partial p_2}{\partial y} \tag{3.2}$$

and the equation of continuity in the form

$$\frac{\partial \rho_2}{\partial t} = \rho_{20}\frac{\partial v_2}{\partial y}, \tag{3.3}$$

to which we must also add a relation between the changes in pressure and density, taken from the equation of state of the medium and the first law of thermodynamics:

$$dp_2 = \frac{p_{20}}{Ic_v T_0} dQ + a_{20}^2 d\rho_2 \tag{3.4}$$

where $c_v$ is the heat capacity at constant volume.

Neglecting terms in $dQ$ which are connected with the heat exchange between different layers of the gas, we assume that

$$dQ = bv_4(t-\tau)_{y=0}\, dt \tag{3.5}$$

($\tau$ is the phenomenological delay in burning).

Replacing the material derivatives in Eq. (3.4) by

$$\left(\frac{\partial p_2}{\partial t} - v_{20}\frac{\partial p_2}{\partial y}\right)dt \text{ and } \left(\frac{\partial \rho_2}{\partial t} - v_{20}\frac{\partial \rho_2}{\partial y}\right)dt,$$

respectively, and neglecting the convection terms, we obtain

$$\frac{\partial p_2}{\partial t} = \frac{p_{20}}{Ic_v T_0} bv_4(t-\tau)_{y=0} + a_{20}^2\frac{\partial \rho_2}{\partial t}. \tag{3.6}$$

Eliminating the density $\rho_2$ from Eqs. (3.3) and (3.6), we finally obtain

$$\frac{\partial p_2}{\partial t} - a_{20}^2\rho_{20}\frac{\partial v_2}{\partial y} = Nv_4(t-\tau)_{y=0}, \tag{3.7}$$

$$N \equiv (p_{20}/Ic_v T_0)\, b.$$

We now write the boundary conditions for Eqs. (3.1), (3.2) and (3.7), and the conditions for connecting the solutions in regions 1,2 and 3:

$$\begin{aligned} &\text{for } y = -L_1: \; v_1 = \nu_1 p_1; \\ &\text{for } y = 0: \; v_4 = \nu_4(p_4 - p_2); \\ &\text{for } y = -L_2: \; v_1 = v_2; \\ &\text{for } y = 0: \; v_2 = v_3; \\ &\text{for } y = L_3: \; v_3 = -\nu_3 p_3; \\ &\text{for } y = L_4: \; p_4 = 0; \\ &\text{for } y = -L_2: \; p_1 = p_2; \\ &\text{for } y = 0: \; p_2 = p_3; \end{aligned} \tag{3.8}$$

$$\nu_1 = \frac{\alpha_1^2}{v_{10}\rho_{10}}, \quad \nu_3 = \frac{\alpha_3^2}{v_{30}\rho_{30}}, \quad \nu_4 = \frac{\alpha_4^2}{v_{40}\rho_{40}}.$$

We can now go on to the basic problem of the investigation of the conditions for self-excitation of the vibration. Following reference 5, we must first of

* The singing flame has also been observed experimentally in a horizontal pipe without a convective air flow[6]

[6] Z. Carrière, Revue d'Acoustique 4, 149 (1953)

all establish the characteristic quasi-polynomial of of the system of equations (3.1), (3.2), (3.7) with the boundary conditions (3.8). That is, we seek solutions of Eqs. (3.1), (3.2), and (3.7) in the form

$$p_i(y, t) = P_i(y) e^{zt}, \tag{3.9}$$
$$v_i(y, t) = V_i(y) e^{zt} \quad (i = 1, 2, 3, 4).$$

After substitution in the corresponding equation for $i = 1$, 3 and 4, we find that

$$P_i(y) = A_i e^{-zy/a_i} + B_i e^{zy/a_i}; \tag{3.10}$$
$$V_i(y) = \frac{1}{a_i \rho_i}[-A_i e^{-zy/a_i} + B_i e^{zy/a_i}],$$

where $A_i$ and $B_i$ are arbitrary constants.

From Eqs. (3.2) and (3.7), we get, after similar substitutions (employing the expression just now obtained for $v_4$),

$$zV_2 = \frac{1}{\rho_{20}} P_2', \tag{3.11}$$
$$zP_2 - a_{20}^2 \rho_{20} V_2' = \frac{N}{a_4 \rho_4}[-A_4 + B_4] e^{-\tau z},$$

whose general solution is

$$P_2(y) = A_2 e^{-zy/a_2} + B_2 e^{zy/a_2} \tag{3.12}$$
$$+ \frac{N}{a_4 \rho_4 z}(B_4 - A_4) e^{-\tau z},$$
$$V_2(y) = \frac{1}{a_2 \rho_2}\{-A_2 e^{-zy/a_2} + B_2 e^{zy/a_2}\}.$$

Now, substituting these solutions in the boundary conditions (3.8), we get a system of eight equations in the arbitrary constants $A_i$, $B_i$, and the frequency $z$. For the sake of simplicity, we shall consider $a_{i0}$ and $\rho_{i0}$ to be single-valued in zones 1, 2 and 3, equal, respectively, to $a$ and $\rho$:

$$-A_1 e^{zL_1/a} + B_1 e^{-zL_1/a} \tag{3.13}$$
$$= \nu_1 a \rho (A_1 e^{zL_1/a} + B_1 e^{-zL_1/a}),$$
$$-A_3 e^{-zL_3/a} + B_3 e^{zL_3/a}$$
$$= -\nu_3 a \rho (A_3 e^{-zL_3/a} + B_3 e^{zL_3/a}),$$
$$-A_4 + B_4 = a_4 \rho_4 \nu_4 (A_4 + B_4 - A_3 - B_3),$$
$$A_4 e^{-zL_4/a_4} + B_4 e^{zL_4/a_4} = 0,$$
$$-A_1 e^{zL_2/a} + B_1 e^{-zL_2/a} = -A_2 e^{zL_2/a} + B_2 e^{-zL_2/a},$$
$$A_1 e^{zL_2/a} + B_1 e^{-zL_2/a} = A_2 e^{zL_2/a}$$
$$+ B_2 e^{-zL_2/a} + \frac{N(B_4 - A_4)}{a_4 \rho_4 z} e^{-\tau z},$$
$$-A_2 + B_2 = -A_3 + B_3,$$
$$A_2 + B_2 + \frac{N(B_4 - A_4)}{a_4 \rho_4 z} e^{-\tau z} = A_3 + B_3.$$

We introduce the notation

$$2\frac{L_1 - L_2}{a} = \tau_1, \quad 2\frac{L_2}{a} = \tau_2, \quad 2\frac{L_3}{a} = \tau_3, \quad 2\frac{L_4}{a} = \tau_4,$$

$$N\nu_4 = 2\mu, \quad \nu_1 a \rho = \mu_1, \quad \nu_3 a \rho = \mu_2, \quad \nu_4 a_4 \rho_4 = \mu_3.$$

Then the characteristic equation of the problem, obtained from Eq. (3.13) by elimination of the constants $A_i$, $B_i$, can be expressed in the following form:

$$\mu_3 \, \mathrm{th} \frac{z\tau_4}{2} = 1 \tag{3.14}$$
$$-\frac{\mu}{z} e^{-\tau z} \frac{(1 + D_2 e^{z\tau_3})(1 + D_1 e^{z(\tau_1 + {}^1/_2 \tau_2)})(e^{z\tau_2/2} - 1)}{1 - D_1 D_2 e^{z(\tau_1 + \tau_2 + \tau_3)}}.$$

Here

$$D_1 = (1 + \mu_1)/(1 - \mu_1), \quad D_2 = (1 + \mu_2)/(1 - \mu_2).$$

We consider a $D$-partition in the parameters $\mu_3$ and $\tau_4$. The boundary of the octant $\mu_3 > 0$, $\tau_4 > 0$ for these parameters consists of the series of curves

$$\mu_3 = -iF(i\omega_j)\,\mathrm{ctg}\,(\omega_j \tau_4/2),$$

where $\omega_j$ are the roots of the equation $\mathrm{Re}F(i\omega_j) = 0$ and $F(z)$ denotes the right hand side of Eq. (3.14). To find the roots of this equation,

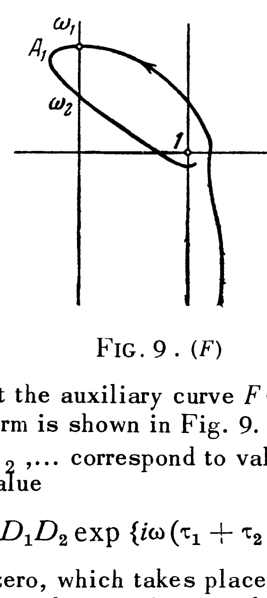


FIG. 9. ($F$)

we construct the auxiliary curve $F(i\omega)$. Its approximate form is shown in Fig. 9. The projections $A_1$, $A_2$, ... correspond to values of $\omega$ for which the value

$$1 - D_1 D_2 \exp\{i\omega(\tau_1 + \tau_2 + \tau_3)\}$$

is close to zero, which takes place (approximately) for frequencies close to the eigenfrequencies of the resonator. The magnitudes of these projections decreases in general, because of the presence of $z$ in the denominator of $F(z)$. Furthermore, some of these can decrease or even disappear:

a) if $1 + D_2 e^{z\tau_3} \approx 0$, i.e., if the top of the tube which supplies the gas lies at a pressure node;

b) if $1 + D_1 \exp \{z[\tau_1 + (\tau_2/2)]\} \approx 0$, i.e., if the center of the burning zone is at a pressure node;

c) if at these frequencies $\exp[i\omega(\tau_2/2)] \approx 1$, i.e., if an integral number of waves is contained in the burning zone.

A diagram of the regions of stability and instability in terms of the parameters $\mu_3$ and $\tau_4$ is shown in Fig. 10*. To each projection of the auxiliary curve which intersects with the imaginary axis, there corresponds a series of regions of instability. The parameters $\mu_3$ and $\tau_4$ correspond to the parameters $D$ and $\tau_1$ of the previous model, and the difference between the plots of Fig. 7 and Fig. 10 lies only in the values and number of roots $\omega_j$ and the values of the quantities $-iF(i\omega j)$ and $Q(i\omega j)$ corresponding.

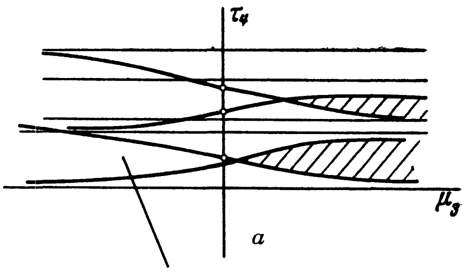


FIG. 10

The new result, on the basis of which we have developed this model, consists of the possibility of exciting not only the base frequency of the resonant tube, but also its higher harmonics, and of the discovery of the dependence of such excitation on the position of the flame, its dimensions and intensity, and the delay in burning. We note that the role of the parameters $\mu$ and $\tau$ for the auxiliary equation in this model and in the preceding one are exactly the same.

The product $D_1D_2$ plays a role here analogous to the damping $h$ in the preceding model; the projections will be increased as $D_1D_2$ approaches unity.

## 4. DISCUSSION OF THE RESULTS

Let us compare the facts observed experimentally by the different authors with the conclusions which follow from the theory presented above. It is natural that the discrete model explains the smallest number of facts, and, conversely, that the distributed model describes the phenomenon most completely.

1. Experiment shows that the vibrations of the singing flame are excited periodically, independent of the length of the supply tube. Model No. 1 does not explain this fact. For models No. 2 and No. 3, this consclusion follows directly from Fig. 7 and Fig. 10. If, for fixed $D(\mu_3)$, $\tau_1$ ($\tau_4$) increases, then the region of stability will be changed to a region of instability, and vice versa. In this case, the excitation conditions can be shown to be quantitatively different from the conditions pointed out by Rayleigh (the flame sounds when the length of the gas tube is somewhat less than $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$,..., and is silent when its length is approximately $\frac{1}{2}\lambda$, $\lambda$, $3/2\,\lambda$,.., where $\lambda$ is the wavelength in the gas). This deviation of the excitation conditions from the conditions of Rayleigh has been observed by several experimenters. The action of progressive waves was made clear in reference 1. These waves apparently arise in a gas tube along with standing waves. As can be seen from the behavior of the curves in Figs. 7 and 10, at sufficiently large $\tau_1$ ($\tau_4$), the system will always be unstable.

2. Experiment shows that the frequency of the singing flame is close to the fundamental eigenfrequency of the sounding tube. However, higher harmonics are sometimes excited. In the latter case, the frequency that is excited depends on the position of the flame in the sounding tube: a short flame is better for the excitation of the higher harmonics. Models No. 1 and No. 2 cannot explain these factors. Model No. 3 does explain them (in model No. 3, the dimensions of the flame are taken into consideration, directly, through the parameter $\tau_2$, and indirectly, by the parameter $\mu$, in which the density of the burning source appears.

3. The system is stable in the absence of flame, when the gas simply blows through the resonator. In this case, the parameter $\mu = 0$. The stability of the system for Model No. 1 can be seen directly from Fig. 4. For models No. 2 and No. 3, the stability results from the fact that the curve $Q(i\omega)$ in Fig. 6 [ $(-iF(i\omega)$ in Fig. 9) ] does not intersect the real axis. The region of stability here occupies the first quadrant ($D > 0$, $\tau_1 > 0$) of Fig. 7 (or, correspondingly, $\mu_3 > 0$, $\tau_4 > 0$, Fig. 10). Experiment indicates that non-hydrogen flames can also sing, for example, the flame of illuminating gas[1,2], but they sing less satisfactorily than the hydrogen flame. This is accounted for by the fact that the parameter $\mu$ is less for the non-hydrogen flame.

4. Experiment shows that if the supply tube is plugged with cotton, so that the gas can still leak through the plug, then the flame will not sing. The effect of the cotton is to increase the damping of the system. For model No. 1, this means an increase in the coefficient $h_2$. The stability of the system increases with rise in $h_2$, as follows from

* The drawings for cases $b$ and $c$ of Fig. 10 are the same as for Figs. 7 $b$ and 7 $c$.

Figs. 3 and 4. For model No. 2, the presence of the cotton means a decrease in the coefficient of consumption $\alpha_1$, i.e., a decrease in the parameters $D$ and $E$. For small $E$ the curve $Q(i\omega)$ does not intersect the real axis, and consequently the system is stable. Similarly for model No. 2, the equation $\mathrm{Im}[-iF(i\omega)] = 0$ does not have real roots for small $\alpha_4$.

5. Experiment shows that the flame does not sing if the opening of the supply tube is very small. For model No. 1, small $S_2$ corresponds to small $\mu\nu$. For small $\mu\nu$, as follows from Fig. 4, the system is stable. For model No. 2, a small area of the aperture $\sigma$ is equivalent to small $E$, and, for model No. 3, to small $\mu$ [the area of the cross section of the gas tube is included in the coefficient $b$ of Eq. (3.5)]. As has already been shown above, the system is stable for small $E$ ($\mu$).

6. Experiment shows that the frequency of the singing flame is close to the eigenfrequency of the resonator, but it can be somewhat different from it (in the case of a large aperture of the supply tube). The proposed theory explains this circumstance by the fact that the frequency $\omega$ from Eq. (1.6), and the values $\omega = \omega_j$, for which $Q(i\omega)$ and $-iF(i\omega)$ intersect the real axis, depend on the parameters of the gas tube and the parameters of burning, as well as on the parameters of the resonator.

7. Since the effect of convection was not considered in the proposed theory, none of the models explain the experimental fact that the flame, when placed in the upper part of the tube, excites vibrations more strongly than when it lies in the lower part of the tube.

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Translated by R. T. Beyer
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