Multiple Scattering in a Coulomb Field in Very Thin Layers of Material

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Data about multiple Coulomb scattering in foils of optical thickness 1 and 3 are given. The relative role of single and triple scattering for different angles of deflection is clarified. In the Appendix a model of Coulomb scattering through small angles is considered.

IN many nuclear experiments it is essential to take into account multiple scattering of charged particles traversing layers of material which are so thin that the entire number of collisions is comparable with unity. The usual approximate theory of multiple scattering, in which it is assumed that the number of collisions is very large, is not applicable here, so that a more exact theory must be employed[1,2] The distribution function for small angles, obtained in this theory, was studied by Biberman[3], but the results of his work cannot be extended to angles, which are large in comparison with the minimum diffraction angle of deflection (see below). Even if the thickness of the scattering foil constitutes one mean free path for elastic scattering, a considerable number of particles undergo more than one collision in traversing it. It is very easy to determine the fraction of such particles in the total scattering effect. With the help of the exact theory of multiple scattering it is also possible to determine the fraction of particles which, after having suffered multiple scattering, leave the material at a definite angle with the original direction of incidence.

The general formula for the distribution function of particles which have passed through some thickness of material and have been deflected through a small angle, was given in references 1 and 2. This function can be written thus:

$$f(\theta) = \frac{1}{2\pi}\int_0^\infty J_0(u\theta)\exp\left\{-N\int d\sigma' \times (1 - J_0(u\theta'))\right\} u\,du, \tag{1}$$

where N is the number of scattering atoms per square centimeter, J_0 the Bessel function of zero order and $d\sigma'$ the differential cross section.

For light elements it is necessary to substitute the differential cross section for Coulomb scattering into Eq. (1), taking into account the screening:

$$d\sigma = \frac{8\pi Z^2 e^4}{m^2 v^4}\frac{\theta d\theta}{[\theta^2 + (\hbar\alpha/mv)^2]^2}; \tag{2}$$

here Z is the atomic number, v the velocity of the particle, α the screening parameter. A potential field of the form $V = (Ze/r)\,e^{-\alpha r}$ corresponds to Eq. (2).

We introduce the optical thickness of the scatterer (d being the geometrical thickness)

$$\tau \equiv 8\pi NZ^2 e^4 d/\hbar^2\alpha^2 v^2 \tag{3}$$

and the dimensionless screening parameter

$$\nu = \hbar\alpha/2mv. \tag{4}$$

Substituting Eqs. (2), (3) and (4) in the general formula Eq. (1) we obtain the formula for multiple Coulomb scattering in light elements:

$$f(\theta) = \frac{e^{-\tau}}{8\pi\nu^2\tau}\int_0^\infty J_0\left(\xi\frac{\theta}{2\nu\sqrt{\tau}}\right) \times \exp\{\xi\sqrt{\tau}\,K_1(\xi/\sqrt{\tau})\}\,\xi\,d\xi; \tag{5}$$

here K_1 is the Macdonald function. For large optical thicknesses this function can be written in the form

$$K_1\left(\frac{\xi}{\sqrt{\tau}}\right) \approx \frac{\sqrt{\tau}}{\xi} - \frac{\xi}{2\sqrt{\tau}}\ln\frac{0.89\xi}{\sqrt{\tau}}, \tag{6}$$

since then the main contributions to the integral in Eq. (5) are given by values of ξ much smaller than $\sqrt{\tau}$

Substitution of Eq. (6) in Eq. (5) yields a formula, valid for large τ (of the order of several tens):

[1] A. S. Kompaneets, J. Exper. Theoret. Phys. USSR 15, 236 (1945)

[2] A. S. Kompaneets, J. Exper. Theoret. Phys. USSR 17, 1059 (1947)

[3] L. M. Biberman, Izv. Akad. Nauk SSSR, Ser. Fiz. 15, 424 (1951)

$$f(\theta)=\frac{1}{8\pi\nu^2\tau}\int\limits_0^{\xi<2\sqrt{\tau}}\exp\left\{\frac{\xi^2}{2}\ln\frac{0{,}89\xi}{\sqrt{\tau}}\right\} \qquad (7)$$
$$\times J_0\left(\xi\frac{\theta}{2\nu\sqrt{\tau}}\right)\xi\,d\xi.$$

A similar expression was obtained in reference 1, where the scattering in heavy elements was investigated, and it turned out that $f(\theta)$ is very reminiscent of the Gaussian distribution. Equation (5) was tabulated by Snyder and Scott[5] for $\tau = 100$ without transition to the asymptotic form (7).

If $\tau = 1$ or 3, Eq. (7) is, of course, entirely inapplicable. To find the distribution formula in this case it is necessary to integrate the Eq. (5) numerically . As was already mentioned, a calculation of $f(\theta)$ for small τ was carried out by Biberman. He replaced the elementary scattering law by a sum of Gaussian functions. Such a sum can give a good of Gaussian functions. Such a sum can give a good approximation to $f(\theta)$ for the smallest deflection angles ($\theta\sim 2\nu$), but it is known not to be applicable in the range $\theta \gg 2\nu$, because the mean square of the deflection angle, as calculated according to Eq. (2), diverges logarithmically, whereas it converges, if Gaussian functions are used instead of Eq. (2). But as the characteristic peculiarity of the Coulomb scattering consists just in the fact that the mean square of the deflection angle diverges on the side of large angles, it follows that in order to investigate $f(\theta)$ for as large θ as possible, one must not use the Gaussian approximation.

In the first place, we note that a direct numerical integration according to Eq. (5) is not feasible, because the integral converges very slowly at the upper limit. Therefore it is necessary to separate out of the integrand the part due to unscattered and singly scattered particles. The remaining distribution function has the form

$$f_{>1}(\theta)=\frac{1}{8\pi\nu^2\tau}e^{-\tau}\int\limits_0^{\infty}J_0\left(\xi\frac{\theta}{2\nu\sqrt{\tau}}\right) \qquad (8)$$
$$\times\left(\exp\left\{\xi\sqrt{\tau}K_1\left(\frac{\xi}{\sqrt{\tau}}\right)\right\}\right.$$
$$\left.-1-\xi\sqrt{\tau}K_1\left(\frac{\xi}{\sqrt{\tau}}\right)\right)\xi\,d\xi.$$

The subscript >1 indicates that only multiple scattering has been taken into account. It is easy to verify that the third term in the bracket under the integral sign gives single scattering, because (see Watson[4])

$$\int\limits_0^{\infty}K_1(ax)J_0(bx)x^2dx=\frac{2a}{(a^2+b^2)^2}.$$

In the case $\tau = 3$ it was also convenient to separate out the double scattering, since the integral (8) does not converge sufficiently well for them. The distribution of doubly scattered particles, written separately, has the form

$$f_2(\vartheta)=\frac{\tau^2e^{-\tau}\vartheta^2}{2\pi(\vartheta^4+4\vartheta^2)^2}\left[\frac{2(\vartheta^2+1)}{(\vartheta^4+4\vartheta^2)^{1/2}}\right. \qquad (9)$$
$$\left.\times\ln\frac{\vartheta^2+2+\sqrt{\vartheta^4+4\vartheta^2}}{\vartheta^2+2-\sqrt{\vartheta^4+4\vartheta^2}}+\vartheta^2-2\right],$$

where

$$\vartheta=\theta/2\nu\sqrt{\tau}. \qquad (10)$$

It can be seen that

$$2\pi\int\limits_0^{\infty}f_2(\vartheta)\,\vartheta d\vartheta=\frac{\tau^2e^{-\tau}}{2},$$

as it should be.

We give the results of numerical computations for $\tau = 1$ and $\tau = 3$ (the calculations were carried out by T. N. Shatalova):

TABLE I

Distribution function for particles which have traversed a foil of thickness of one mean free path ($\tau = 1$)

$\vartheta=\frac{\theta}{2\nu\sqrt{\tau}}$	0	0.5	1	1.5	2	2.5
$2\pi f_{>1}$	0.149	0.130	0.091	0.057	0.0342	0.0204
$2\pi f_1$	0.735		0.184		0.0394	

ϑ	3	3.5	4	4.5	5
$2\pi f_{>1}$	0.0136	0.00735	0.00515	0.00386	0.0026
$2\pi f_1$	0.00735		0.00254		0.0011

Here $f_{>1}(\theta)$ denotes the distribution function of particles which have suffered more than one col-

[5] H. S. Snyder and V. T. Scott, Phys. Rev. **76**, 220 (1949)

[4] T. Watson, *Bessel Functions*

lision, f_1 the distribution function of particles which have suffered one collision. It turns out that the number of the former and the number of the latter become comparable to $\theta = 2$, and for larger angles the particles which have suffered multiple scattering prevail.

The distribution given in Table 1 is normalized to unity:

$$e^{-1} + 2\pi \int_0^\infty (f_{>1} + f_1)\, \vartheta d\vartheta = 1.$$

Here e^{-1} is the fraction of particles which have passed without a single scattering event. Actually Table 1 comprises 97.5% of all particles which have passed; the remaining particles are scattered through angles θ larger than 5.

TABLE II

Distribution function for particles which have traversed a foil of thickness of three mean free paths ($\tau = 3$)

$\vartheta = \frac{\theta}{2\nu\sqrt{\tau}}$	0	0.5	1.0	1.5	2.5	4	6	8
$2\pi f_{>1}(\vartheta)$	0.295	0.268	0.205	0.143	0.0645	0.0201	0.0043	0.0023
$2\pi f_1(\vartheta)$	0.298	0.191	0.746	0.028	0.0118	0.00103	0.0002	0.00007
$2\pi f_2(\vartheta)$	0.149	0.133	0.088	0.056	0.0162	0.030	0.0075	0.00024

This table comprises 92% of all particles which have passed. We note that now multiple scattering becomes larger than single scattering already for very small values of θ.

APPENDIX

The distribution function for multiple Coulomb scattering can also be studied experimentally with the help of a simple optical model.

Consider an emulsion compounded of two transparent media of very close refractive indices. We will assume that one of the substances constitutes the medium proper, into which the other has been injected in the form of small spheres of equal radius R, distributed at random. This radius will be assumed to be large in comparison with the wave length of the transmitted light, so that the laws of geometrical optics can be applied. Thus the scattering under consideration is, in a certain sense, opposite to the Rayleigh scattering, where the wave length is large as compared with the dimensions of the obstacle.

We will first determine the deflection of a light ray in an elementary scattering event. According to assumption, the relative refractive index ν of the two media is a small number. We draw the polar axis from the center of the sphere in the direction of incidence of the ray. The polar angle of the point of incidence of the ray on the sphere will be called β. Then, as is seen from an elementary construction, the deflection angle of the ray in the sphere is

$$\theta = 2\nu \tan\beta.$$

The angle θ is regarded as small, since $\nu \ll 1$, and the contribution from larger angles is insignificant. From this it is easy to obtain the differential scattering cross section. According to general formulas of the classical scattering theory it is equal to*

$$d\sigma = 2\pi R^2 \frac{4\nu^2\theta d\theta}{(4\nu^2 + \theta^2)^2}.$$

This formula is quite similar to Eq. (2), with the designation ν corresponding to the previous one, and

$$R = 2Zc^2\ h\alpha v.$$

The formula for $d\sigma$ can also be obtained from the diffraction theory by a transition to the limit.

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* We notice that Eq. (2) is obtained in the Born approximation from the wave equation, and the present formula in the approximation of geometrical optics. The wave aspect for particles appears to be analogous to the ray aspect for light. Consequently the analogy found here is by no means a special case of the optical-mechanical analogy. We were unable to find out the physical reason for this analogy.