The Statistical Theory of Heavy Nuclei and of Nuclear Forces

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A statistical method representing a semi-classical approximation employing the self-consistent field method is developed for application to heavy nuclei. The method takes into consideration unfilled spin and charge states for various charge-symmetry functions of interaction between nucleons with the separation dependence having the form $-g^2e^{-\kappa r}/r$. Equations have been derived and analyzed both for the case of completely filled state of nucleons as well as for the "charge" or "spin" states. Formulas have been obtained for the basic "isotopic" and supplementary term of the equation expressing the energy of a nucleus. A discussion is included regarding the influence of various properties of nuclear forces on the behavior of a complex nuclei.

INTRODUCTION

THE investigation of a heavy nuclei by a statistical method which takes into account the properties of nuclear forces, predicted by theory and qualitatively substantiated for a system consisting of two nucleons, is of interest both for the explanation of properties of heavy nuclei, as well as for the study of nuclear interactions. A number of recent papers have been devoted to the study of nuclei using statistical methods. Ivanenko and others[1,2] used the methods of Thomas-Fermi in their analysis of nuclear shells. The nuclear potentials were considered to be functions of distance $f(r)$ of the type of Yukawa potential (without exchange and independent of spin). In the work of Kompaneets[3] the self - consistent method was applied to the study of a nucleus with saturation spins and charges, with an interaction function, representing one half the sum of ordinary exchange forces, having the same co-ordinate dependence $f(r)$.

In this paper a derivation for the statistical theory of nucleus is presented, which permits analysis of both the case of the saturation of spins and charges and the case where one or the other is incomplete for various charge-symmetry functions of interaction between nucleons (without regard to tensor forces). The solution obtained is in a general form in the sense that the derived formulas contain coefficients which depend upon the form of operator of the function of interaction, and therefore yield concrete results. An expression has been derived for the energy of a nucleus which takes into account the kinetic energy of nucleons the energy of "direct" and "exchange" interaction, as functions of two densities ρ_1 and ρ_2, which represent respectively the density of neutrons and protons under conditions of complete filling spins and incomplete saturation of charges, and densities of particles with spins directed "upward" and "downward", under the conditions of complete saturation of charges and incomplete saturation of spins. On the basis of the variation method and equation for nuclear potentials, a system of equations has been obtained which specifies the distribution of the total density of nucleons $\rho_0 = \rho_1 + \rho_2$ and the charge or spin density $\rho = \rho_1 - \rho_2$. In view of the small magnitude of ρ/ρ_0 equations have been obtained for ρ_0, in the first approximation which are independent of ρ, and which have the same form as they do in the case of complete saturation of spins and charges. In this case ρ is determined by linear differential equations with coefficients dependent upon ρ_0. Study of the equation for ρ_0 along with the equations for energy leads to the selection of interaction functions which insure complete saturation of nuclear forces; the ratio of "exchange" forces to "ordinary" has proven to equal four. By making use of the magnitude of the parameter g^2/κ, determined by the condition of existence of deuteron, the possibility is established that heavy nuclei can exist in stable state having binding energy proportional to A, and radii $R = r_0 A^{1/3}$ where r_0 is roughly equal to the radius of action of nuclear forces. On the basis of these parameters, which determine the interaction function of two nucleons, independently of the exact form of the potential

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[1] D. D. Ivanenko and D. Rodichev, Doklady Akad. Nauk SSSR **70**, 605 (1950)

[2] D. D. Ivanenko and A. A. Sokolov, Doklady Akad. Nauk SSSR **74**, 33 (1950)

[3] A. S. Kompaneetz, Doklady Akad. Nauk SSSR **85**, 301 (1952)

hole, expressions are also derived for "isotopic" and supplementary term $\delta(A, Z)$ of the semi-empirical equation for the energy of a nucleus [Ref. 4, Eq. (1.8)]. With the aid of the supplementary parameter -- having experimental value r_0 -- it is possible to compute coefficients of the various terms of the expression for the energy of nucleus and to estimate the mass of the meson. The parameter γ also appears in the theory, taking into account the dependence of nuclear forces due to spins. It has only a small quantitative effect on the results and is used merely for qualitative deductions made during analysis of distribution of ρ.

1. GENERAL THEORY

Let us examine the function of interaction between two nucleons expressed in the following form:

$$U_{1,2} = P(1,2) f(r_{1,2}), \tag{1}$$

where

$$f(r_{1,2}) = -g^2 (e^{-\varkappa r_{1,2}} / r_{1,2}); \tag{2}$$

P – an operator which can assume any one of the following forms:

$$P = P_0 = 1, \quad P = P_\sigma = \frac{1 + (\vec{\sigma}_1 \vec{\sigma}_2)}{2} \tag{3}$$

$$P = P_\tau = \frac{1 + (\vec{\tau}_1 \vec{\tau}_2)}{2},$$

$$P = P_r = -\frac{1 + (\vec{\sigma}_1 \vec{\sigma}_2)}{2} \frac{1 + (\vec{\tau}_1 \vec{\tau}_2)}{2},$$

or can be a linear combination of these quantities. Operators P_σ, P_τ, P_r correspond, as is known, to rearrangement of spin, charge and space coordinates of the two nucleons. Let us express the energy of a nucleus in the following form:

$$E = \int [T_1(\rho_1) + T_2(\rho_2)] d\tau \tag{4}$$
$$+ \frac{\alpha}{2} \int [A_1(\rho_1) + A_2(\rho_2)] d\tau$$
$$+ \alpha_{1,2} \int A_{1,2}(\rho_1, \rho_2) d\tau$$
$$+ \frac{\beta}{2} \iint [\rho_1(1)\rho_1(2) + \rho_2(1)\rho_2(2)] f(r_{1,2}) d\tau_1 d\tau_2$$
$$+ \beta_{1,2} \iint \rho_1(1)\rho_2(2) f(r_{1,2}) d\tau_1 d\tau_2,$$

where the first term represents kinetic energy:

$$T_1(\rho_1) + T_2(\rho_2) \tag{5}$$
$$= \frac{3}{10}\left(\frac{3}{\pi}\right)^{2/3} \frac{\pi^2 \hbar^2}{M} (\rho_1^{5/3} + \rho_2^{5/3});$$

the second and third terms are "exchange" interactions, respectively, for "like" and "unlike" particles, the fourth and fifth, analogous forms of "direct" interaction. The coefficients α, $\alpha_{1,2}$, β, $\beta_{1,2}$ depend upon the form of operator used in Eq. (1) and are computed in the same manner as the functions A_1, A_2, $A_{1,2}$, i.e., with the aid of the wave function Ψ of nucleus in the form of a determinant composed of the individual wave functions $\varphi_i(\mathbf{X}_i)$ of all N nucleons ($\mathbf{X}_i$ represents the various space co-ordinates, spin and charge co-ordinates of nucleons). It is assumed that

$$\phi(\mathbf{X}_i) = \psi_i(x_i, y_i, z_i)\, \eta_i(\vec{\sigma}_i)\, \zeta(\vec{\tau}_i), \tag{6}$$

where η and ζ are functions only of operators σ_z, τ_z respectively. The energy of interaction of all nucleons can be expressed as follows:

$$U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \iint \varphi_i^*(1) \varphi_j^*(2) U_{1,2} \varphi_i(1) \varphi_j(2) d\mathbf{X}_1 d\mathbf{X}_2 \tag{7}$$
$$- \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \iint \varphi_j^*(1) \varphi_i^*(2) U_{1,2} \varphi_i(1) \varphi_j(2) d\mathbf{X}_1 d\mathbf{X}_2$$
$$= J + K = \frac{1}{2} \sum_{i,j} J_{i,j} + \frac{1}{2} \sum_{i,j} K_{i,j}.$$

Taking a summation over the spin and charge variables in the expressions for $J_{i,j}$ and $K_{i,j}$ from Eq. (7), and taking into account Eq. (6), we express the direct interaction J in Eq. (7) by ordinary densities

$$\rho_1 = \sum_{i=1}^{N_1} |\psi_i|^2, \quad \rho_2 = \sum_{j=1}^{N_2} |\psi_j|^2,$$

exchange interaction K by the mixed densities

$$\rho_1(1,2) = \sum_{i=1}^{N_1} \psi_i(1) \psi_i^*(2), \tag{8}$$

$$\rho_2(1,2) = \sum_{j=1}^{N_2} \psi_j(1) \psi_j^*(2)$$

[4] E. Fermi, *Nuclear Physics*

(N_1, N_2 -- the number of particles of both "types" correspondingly), and introduce the notation:

$$\iint |\rho_1(1,2)|^2 f(r_{1,2})\,d\tau_1\,d\tau_2 = \int A_1(\rho_1)\,d\tau, \qquad (9)$$

$$\iint |\rho_2(1,2)|^2 f(r_{1,2})\,d\tau_1\,d\tau_2 = \int A_2(\rho_2)\,d\tau,$$

$$\iint \rho_1(1,2)\rho_2^*(1,2) f(r_{1,2})\,d\tau_1\,d\tau_2 = \int A_{1,2}(\rho_1,\rho_2)\,d\tau,$$

we derive the values of coefficients given in Table 1 for the four cases of Eq. (3).

In the case in which the operator $P(1,2)$ in Eq. (1) consists of a linear combination of any of the operators of Eq. (3), a situation which corresponds, physically, to the possible forms of interaction, the coefficients α, $\alpha_{1,2}$, β, $\beta_{1,2}$ are evaluated with the aid of the corresponding linear combinations from coefficients of Table 1. In Table 2 are represented the values of coefficients and some other quantities for a number of possibilities considered in the following Sections. In both tables in the column "Type of Saturation" line "a" corresponds to saturation of spins, line "b", to the saturation of charges.

The functions A_1, A_2 and $A_{1,2}$ were computed on the basis of Eqs. (2), (8) and (9), assuming that ψ_i, ψ_j have the form of plane waves, and substituting summation over i, j, by integration in momentum space p_1, p_2; the weight of each state was taken, as in Eq. (5), to be equal to two (two spin states or two charge states of nucleons). This computation is analogous to the computation of the exchange energy of an electron system (Ref. 5, Sec. 2) with the replacement of the Coulomb force by Eq. (2) and with consideration of the fact that the maximum momenta $p_{\mu,1}$, $p_{\mu,2}$ are, in our problem, different for particles of the two "types". Making use of the relation

$$\rho_1 = 2\left(\tfrac{4}{3}\pi p_{\mu,1}^3\right)/(2\pi\hbar)^3 \qquad (10)$$

and analogous relation for ρ_2, we obtain the magnitudes of A_1, A_2, $A_{1,2}$ as functions of densities ρ_1 ans ρ_2:

$$A_{1,2} = -\frac{g^2\varkappa^4}{\pi^3}\left\{\left[\frac{1}{24} + \frac{\varepsilon^4}{4}\left(\rho_1^{2/3}\rho_2^{2/3} - \frac{1}{2}\rho_1^{4/3} - \frac{1}{2}\rho_2^{4/3}\right)\right.\right. \qquad (11)$$

$$\left. + \frac{\varepsilon^2}{4}\left(\rho_1^{2/3} + \rho_2^{2/3}\right)\right] \ln\frac{1+\varepsilon^2(\rho_1^{1/3}+\rho_2^{1/3})^2}{1+\varepsilon^2(\rho_1^{1/3}-\rho_2^{1/3})^2}$$

$$- \tfrac{2}{3}\varepsilon^3\left[(\rho_1+\rho_2)\arctan \varepsilon(\rho_1^{1/3}+\rho_2^{1/3}) - (\rho_1-\rho_2)\right.$$

$$\left.\times \arctan \varepsilon(\rho_1^{1/3}-\rho_2^{1/3})\right]$$

$$\left. - \tfrac{1}{6}\varepsilon^2\rho_1^{1/3}\rho_2^{1/3} + \frac{1}{2}\varepsilon^4\left(\rho_1^{1/3}\rho_2 + \rho_1\rho_2^{1/3}\right)\right\},$$

$$A_1 = -\frac{g^2\varkappa^4}{\pi^3}\left[\left(\frac{1}{24} + \frac{\varepsilon^2}{2}\rho_1^{2/3}\right)\right. \qquad (12)$$

$$\left. \times \ln\left(1 + 4\varepsilon^2\rho_1^{2/3}\right)\right.$$

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[5] P. Gombas, *The Statistical Theory of the Atom and its Applications*

TABLE I

Operator	Type of Saturation	α	$\alpha_{1,2}$	β	$\beta_{1,2}$
$P_0=1$	a, b	$-1/2$	0	1	1
P_r	a, b	1	1	$-1/2$	0
P_σ	a	-1	0	$1/2$	$1/2$
	b	$-1/2$	$-1/2$	1	0
P_τ	a	$-1/2$	$-1/2$	1	0
	b	-1	0	$1/2$	$1/2$

TABLE II

Case No.	Operator	Value for various states of a system of two nucleons: Triplet l - even	Single l - even	Triplet l - odd	Single l - odd	Type of Saturation	α	$\alpha_{1,2}$	β	$\beta_{1,2}$	$\alpha+\alpha_{1,2}$	$\beta+\beta_{1,2}$	$\beta-\beta_{1,2}$
1	$\frac{1+P_r}{2}$	1	1	0	0	a, b	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$
2	$\frac{1+P_r}{2}(1+\gamma P_\sigma) = \frac{1+P_r}{2}+\frac{\gamma}{2}(P_\sigma - P_\tau)$	$1+\gamma$	$1-\gamma$	0	0	a	$\frac{1}{4}-\frac{\gamma}{4}$	$\frac{1}{2}+\frac{\gamma}{4}$	$\frac{1}{4}-\frac{\gamma}{4}$	$\frac{1}{2}+\frac{\gamma}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\left(\frac{1}{4}+\frac{\gamma}{2}\right)$
						b	$\frac{1}{4}+\frac{\gamma}{4}$	$\frac{1}{2}-\frac{\gamma}{4}$	$\frac{1}{4}+\frac{\gamma}{4}$	$\frac{1}{2}-\frac{\gamma}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\left(\frac{1}{4}-\frac{\gamma}{2}\right)$
3	$\frac{1+4P_r}{5}$	1	1	$-\frac{3}{5}$	$-\frac{3}{5}$	a, b	$\frac{7}{10}$	$\frac{4}{5}$	$-\frac{1}{5}$	$+\frac{1}{5}$	$\frac{3}{2}$	0	$-\frac{2}{5}$
4	$\frac{1+4P_r}{5}+\frac{\gamma}{2}(P_\sigma - P_\tau)$	$1+\gamma$	$1-\gamma$	$-\frac{3}{5}$	$-\frac{3}{5}$	a	$\frac{7}{10}-\frac{\gamma}{4}$	$\frac{4}{5}+\frac{\gamma}{4}$	$-\frac{1}{5}-\frac{\gamma}{4}$	$+\frac{1}{5}+\frac{\gamma}{4}$	$\frac{3}{2}$	0	$-\left(\frac{2}{5}+\frac{\gamma}{2}\right)$
						b	$\frac{7}{10}+\frac{\gamma}{4}$	$\frac{4}{5}-\frac{\gamma}{4}$	$-\frac{1}{5}+\frac{\gamma}{4}$	$+\frac{1}{5}-\frac{\gamma}{4}$	$\frac{3}{2}$	0	$-\left(\frac{2}{5}-\frac{\gamma}{2}\right)$

$$-\frac{4}{3}\varepsilon^3\rho_1 \arctan 2\varepsilon\rho_1^{1/3} + \varepsilon^4\rho_1^{4/3} - \frac{1}{6}\varepsilon^2\rho_1^{2/3}\Big],$$

$$\varepsilon = (3/\pi)^{1/3}(\pi/\varkappa); \quad (13)$$

A_2 is analogous to A_1, with ρ_1, substituted for ρ_2. For the case of saturation of spin and charge states $\rho_1 = \rho_2$, and $A_1 = A_2 = A_{1,2}$. Introducing new variables $\rho_0 = \rho_1 + \rho_2$ and $\rho = \rho_1 - \rho_2$ and at the same time decomposing Eqs. (5), (11) and (12) in powers of ρ/ρ_0, ignoring powers higher than the 2nd, we obtain an expression for the energy of nucleus [Eq. (4)] in the following form:

$$E = E_0 + E_I, \quad (14)$$

where

$$E_0 = {}^3/_{10}\, c_0 \int \rho_0^{5/3} d\tau - a_0(\alpha + \alpha_{1,2}) \quad (15)$$

$$\int [(1 + 3x^2)\ln(1 + x^2) - 4x^3 \arctan x$$

$$+ {}^3/_2\, x^4 - x^2]\, d\tau - (g/4)(\beta + \beta_{1,2}) \int \rho_0 V_0\, d\tau,$$

$$E_I = \int \frac{\rho^2}{\rho_0^2}\Big\{\frac{1}{6} c_0 \rho_0^{5/3} - \frac{1}{3} a_0 [(\alpha + \alpha_{1,2}) \quad (16)$$

$$\times (x^4 - x^2 \ln(1 + x^2)) - \alpha_{1,2} x^4 \ln(1 + x^2)]\Big\} d\tau$$

$$-(g/4)(\beta - \beta_{1,2}) \int \rho V\, d\tau,$$

$$x = 2\pi\left(\frac{3}{2\pi}\right)^{1/3} \frac{\rho_0^{1/3}}{\varkappa}, \quad (17)$$

$$c_0 = \left(\frac{3}{2\pi}\right)^{2/3} \frac{\pi^2\hbar^2}{M}, \quad a_0 = \frac{g^2\varkappa^4}{24\pi^3}, \quad (18)$$

$$V_0(\mathbf{r}) = g \int \rho_0(\mathbf{r}') \frac{\exp\{-\varkappa\,|\mathbf{r} - \mathbf{r}'|\}}{|\mathbf{r} - \mathbf{r}'|} d\tau, \quad (19)$$

V is analogous to V_0, with $\rho_0(\mathbf{r}')$ substituted for $\rho(\mathbf{r}')$.

Making use of the variation method, the equation for ρ_0 and ρ is found from the condition

$$\delta E(\rho_0, \rho) = 0 \quad (20)$$

with the supplementary conditions

$$\int \rho_0\, d\tau = N = A = \text{const}, \quad (21a)$$

$$\int \rho\, d\tau = I = \text{const}, \quad (21b)$$

where I is the isotopic number $A - 2Z$, if ρ represents the charge density, and spin number $2s$ (s - the resultant spin in the components of $\hbar$) if ρ represents the spin state density.

On the basis of Eqs. (14) - (21) we obtain the following system of equations

$$\frac{1}{2} c_0 \rho_0^{2/3} - (\alpha + \alpha_{1,2}) \frac{2a_0}{\rho_0} [x^4 \quad (22a)$$

$$+ x^2 \ln(1 + x^2) - 2x^3 \arctan x]$$

$$-\frac{g}{2}(\beta + \beta_{1,2}) V_0 - \frac{1}{9}\frac{\rho^2}{\rho_0^2}\Big\{\frac{1}{2} c_0 \rho_0^{2/3}$$

$$-2\frac{a_0}{\rho_0} x^4 \left[\alpha\left(\frac{2 + x^2}{1 + x^2} - 2\frac{\ln(1 + x^2)}{x^2}\right)\right.$$

$$\left.\left. + \alpha_{1,2}\left(2 - 2\frac{\ln(1 + x^2)}{x^2} - \ln(1 + x^2)\right)\right]\right\} = -\lambda_0,$$

$$\frac{\rho}{\rho_0}\Big\{\frac{1}{3} c_0 \rho_0^{2/3} - \frac{2}{3}\frac{a_0}{\rho_0} x^4 \Big[(\alpha + \alpha_{1,2}) \quad (22b)$$

$$\times \left(1 - \frac{\ln(1 + x^2)}{x^2}\right) - \alpha_{1,2} \ln(1 + x^2)\Big]\Big\}$$

$$-\frac{g}{2}(\beta - \beta_{1,2}) V = -\lambda,$$

which must be solved simultaneously with the differential equations for nuclear potentials V_0 and V. Here λ_0 and λ are Lagrange's multipliers corresponding to the Eqs. (21a) and (21b).

2. NUCLEI WITH SATURATED SPINS AND CHARGES

The density distribution of nucleons ρ_0 is determined in this case ($\rho = 0$), on the basis of Eqs. (19) and (22a), by the following system of equations:

$$c_0 \rho_0^{2/3} - \frac{4a_0}{\rho_0}(\alpha + \alpha_{1,2})[x^4 + x^2 \ln(1 + x^2) \quad (23)$$

$$- 2x^3 \arctan x] - (\beta + \beta_{1,2}) g V_0 = -2\lambda_0,$$

$$\Delta V_0 - \varkappa^2 V_0 = -4\pi g \rho_0, \quad (24)$$

where V_0 must satisfy the condition of finiteness at the center of the nucleus and condition

$$(d/dr)[\ln(rV_0)]_{r=R} = -\varkappa$$

(R - radius of nucleus). The solution of this system depends upon the values of coefficients $\alpha + \alpha_{1,2}$ and $\beta + \beta_{1,2}$. On the basis of Eqs. (23) and (21a), Eq. (15) for the energy E_0 assumes the following form:

$$E_0 = \frac{1}{20} c_0 \int c_0^{5/3} d\tau - (\alpha + \alpha_{1,2}') a_0 \tag{25a}$$

$$\int \varphi(x) d\tau - \frac{\lambda_0}{2} A,$$

where

$$\varphi(x) = (1 + 2x^2)\ln(1 + x^2) - 2x^3 \arctan x + {}^1/_2 x^4 - x^2. \tag{25b}$$

Lagrange's multiplier λ_0 is connected with E_0 by the relation

$$\lambda_0 = -(\partial E/\partial A)_I = -(dE_0/dA) \tag{26}$$

and represents the binding energy of one nucleon. If λ_0 does not depend on A, the effect of saturation takes place: $E_0 = -\lambda_0 A$, and it is this specific condition which limits the choice of the function of interaction. It is easy to see that saturation, without any doubt, takes place for forces which lead to the disappearance of direct interaction in Eq. (15), i.e., in the case $\beta + \beta_{1,2} = 0$. Actually, in this case, the term which contains V_0 in Eq. (23) also disappears, and ρ_0, determined by this equation, is constant within the bounds of the nuclei (the boundary conditions for V_0, on the basis of Eq. (24) and with constant density ρ_0 within a nucleus of radius R, with $\rho_0 = 0$ with $r > R$, are satisfied automatically here. From Eqs.(25a) and with the aid of (26), with ρ_0 = const, we obtain the following equation:

$$\lambda_0 = -\frac{1}{10} c_0 \rho_0^{2/3} + 2(\alpha + \alpha_{1,2}) \frac{a_0}{\rho_0} \varphi(x), \tag{27}$$

Here, on the basis of Eq. (17),

$$x = 3(\pi/3)^{1/3} (1/\varkappa r_0), \tag{28}$$

where r_0 – radius of the space belonging to one nucleon. From Eq. (23), with $\beta + \beta_{1,2} = 0$, and from Eq. (27), taking into account Eqs. (28) and (18), it is possible to determine λ_0 and r_0 through the constants g, κ, $a + a_{1,2}$; therefore λ_0 and r_0 do not depend on A in this case, which characterizes the effect of saturation. Excluding λ_0 from Eqs. (23), (27), and taking into consideration Eq. (28), we obtain the following equation:

$$x^3 = \frac{10}{\pi} \frac{M}{\hbar^2} \frac{g^2}{\varkappa} (\alpha + \alpha_{1,2}) f(x), \tag{29a}$$

where

$$f(x) = 1 + \frac{x^2}{2} - \ln(1 + x^2) - \frac{\ln(1 + x^2)}{x^2}. \tag{29b}$$

The root of this equation X_0, and therefore also the magnitude of $1/\kappa r_0$ depend not on the exact value of g and κ, but merely on the magnitude of g^2/κ which on the basis of theory of deuteron is approximately constant.

With the aid of relation $g^2/\kappa \approx \pi^2\hbar^2/4M$, and employing operator 3, Table 2 (which satisfies the conditions $\beta + \beta_{1,2} = 0$), we find $\chi_0 = 5$; taking operator 4, likewise satisfying this condition, with $\gamma \approx 0.15$, $(1 + \gamma)(g^2/\kappa) = (g^2/\kappa)_{tp} \approx \pi^2\hbar^2/4M$, we obtain $\chi_0 \approx 4$. On the basis of Eq. (28) an approximate equality is obtained for the radius of action of nuclear forces $1/\kappa$ and the radius of space available for one nucleon, r_0, which agrees with the experiment. The values of χ_0 determined in this manner, assure a positive value of λ_0 which can be computed on the basis of Eqs. (27) - (29) from the relation

$$\lambda_0 = \frac{3\pi}{10} \left(\frac{3}{\pi}\right)^{1/3} \frac{\hbar^2}{M} \frac{1}{r_0^2} \left[\frac{\varphi(x_0)}{x_0^2 f(x_0)} - \frac{1}{4}\right]. \tag{30}$$

With $\chi_0 = 4$, $r_0 = 1.5 \times 10^{-13}$ cm we obtain for λ_0 the value of approximately 3 MeV. At that, from Eq. (28) $1/\kappa \approx 2 \times 10^{-13}$ cm, which corresponds to the mass of a meson $\mu \approx 200\, m_e$. The experimental value $\lambda_0 \approx 8$ MeV is obtained from Eq.(30) with $r_0 \approx 0.9 \times 10^{-13}$ cm. It should be noted that the magnitude of r_0 computed in this manner must actually be somewhat smaller than the experimental value, as a result of ignoring not only Coulomb forces of isotropic and surface effects but also the peculiarities of the method of variation.

It must also be pointed out that for the forces we have considered, which lead to saturation [i.e., to the disappearance of the term containing V_0 in Eqs. (15), (23)], the condition $\partial E_0/\partial R = 0$ is automatically satisfied, thus insuring stability of the surface of the nucleus.

Let us now examine the question regarding possible existence of a solution to the system of Eqs.(2 3) and (24) with $\beta + \beta_{1,2} \neq 0$ which gives approximately constant ρ_0 in the major part of volume, i.e., the solution, the zero approximation of which, within the nucleus (away from the shell), has the form:

$$\rho_0 = \text{const} = \frac{1}{{}^4/_3 \pi r_0^3}, \quad V_0 = \frac{4\pi g \rho_0}{\varkappa^2}. \tag{31}$$

If such a solution is possible then Eq. (27) remains roughly correct and with the condition (31) from Eqs. (23) and (27) we obtain the generalization (29) in the form

$$x^3 = \frac{10}{\pi}\frac{M}{\hbar^2}\frac{g^2}{\varkappa}\left[(\beta + \beta_{1,2})\frac{x^4}{6} + (\alpha + \alpha_{1,2})\, f(x)\right]; \quad (32)$$

which determines the equality $\chi = \chi_0$.

If operators 1, 2 (Table 2) are chosen, the solution of Eq. (32) is obtained in the form : $\chi = \chi_0 \approx 0.6$. From this $(1/\kappa)$: $r_0 \approx 0.2$. Also λ turns out to be negative, i.e., this solution does not correspond to a stable state of the nuclei. On the basis of general considerations regarding the role of negative direct interaction and also on the basis of investigation of the polarity of second differential of E_0, we also come to the conclusion that it is impossible to have stable states of nuclei under the existence of forces whose expressions contain operators of type 1, 2, and which are saturated. The same applies to forces examined in Ref. 3, which differ from forces corresponding to operators of type 1 (Table 2) by the polarity of their exchange operator and which lead, therefore, to positive energy of exchange and to disappearance of the entire interaction for the condition of two nucleons with even l.

3. NUCLEI WITH UNSATURATED SPINS AND CHARGES

The distribution of spin or charge density ρ on the basis of Eq. (22b) is determined by the expression:

$$\rho = \frac{(g/2)\,(\beta - \beta_{1,2})\, V - \lambda}{K(\rho_0)}, \quad (33)$$

where $K(\rho_0)$ is the expression in curly brackets in Eq. (22b), divided by ρ_0; moreover, V is related to ρ by the equation

$$\Delta V - \varkappa^2 V = -4\pi g \rho. \quad (34)$$

From Eqs. (33) and (34) we obtain the equation for potential V:

$$\Delta V - \mu^2 V = L, \quad (35)$$

where

$$\mu^2 = \varkappa^2 - 2\pi g^2 \frac{\beta - \beta_{1,2}}{K(\rho_0)}, \quad (36a)$$

$$L = 4\pi g \lambda / K(\rho_0), \quad (36b)$$

from Tables 1 and 2, K and μ^2 are positive for ordinary, exchange and "mixed" forces.

Assuming further that ρ_0 is approximately constant within the nucleus, we obtain solutions to Eq. (34) which are spherically-symmetrical and regular at the origin, of the form:

$$V = C\frac{\operatorname{sh}\mu r}{r} - \frac{L}{\mu^2}. \quad (37)$$

We then get, from Eqs. (33) and (36),

$$\rho = -\left(\frac{B \operatorname{sh}\mu r}{r} + \frac{1}{K}\frac{\varkappa^2}{\mu^2}\right)\lambda. \quad (38)$$

Constant C and the related constant B are determined from boundary condition

$$\left[\frac{d}{dr}\ln\,(rV)\right]_{r=R} = -\varkappa. \quad (39)$$

This yields

$$B = \frac{-(\beta - \beta_{1,2})\, 2\pi g^2}{K^2\mu^2}\,\frac{\varkappa R + 1}{\varkappa \operatorname{sh}\mu R + \mu \operatorname{ch}\mu R}. \quad (40)$$

On the basis of Eqs. (21a), (21b) and also (38) and (40), we obtain the follow'ng expression for λ:

$$\lambda = -\rho_0 KI/A\left[\left(1 - \frac{\varkappa^2}{\mu^2}\right)F(R) + \frac{\varkappa^2}{\mu^2}\right] = -b\frac{I}{A}, \quad (41)$$

where

$$F(R) = \frac{3}{(\mu R)^2}(1 + \varkappa R)\,\frac{1 - (\operatorname{th}\mu R\,/\,\mu R)}{1 + (\mu/\varkappa)\operatorname{th}\mu R}. \quad (42)$$

λ is negative since, for all forces of interest, $\mu^2 \geq \kappa^2$, $\mu R \gg 1$.

Further, taking into account the values of coefficients $\beta - \beta_{1,2}$ for the different cases(from Tables 1 and 2), we note that B is positive for exchange and "mixed" forces and is equal to zero for ordinary forces; therefore, Eq. (38) for distribution of ρ contains a term (determined by exchange forces) which increases from the center of the nucleus toward the periphery (with $\rho_0 \approx$ const). This means that "excess" neutrons in a nucleus with unsaturated charges and saturated spin states experience additional mutual repulsion; and similarly, mutual repulsion exists between "excess" nucleons with parallel spins in the case of saturation of charges and unsaturated spin states.

We wish to point out that the above effect of additional repulsion does not disappear even in the presence of spin dependence(for example, for the case of forces corresponding to operator 4, Table 2), as a result of the small magnitude of γ. However, the effect diminishes for the case of unsaturated spin states and saturation of charges and increases in the case of unsaturated charges and saturation of spin states because of the "spin" terms. This means that, for example, for a nucleus of type $A = 4n + 2$, which is of great importance from the energy point of view, (if Coulomb forces and differences of mass of proton and neutron are ignored), a state exists with unsaturated spin states and saturated charges, which corresponds to a singlet charge state and triplet spin state with two "excess" nucleons (proton and neutron with parallel spin) in the shell model.

In this manner, the statistical model, as well as the free particle model, can explain the existence of nuclei of type "Z - odd, A - even" with spin equal to unity, by taking into account spin interaction.

The expression for E_I can be studied in more detail on the basis of the newly found distribution of ρ.

On the basis of Eqs. (16), (22b), (21b), we obtain

$$E_I = -(\lambda/2)\, I, \qquad (43)$$

from which, taking into account the value of λ from Eq. (41), we obtain

$$E_I = +\frac{1}{2}\frac{\rho_0 K}{[1-(\varkappa^2/\mu^2)]\,F(R) + (\varkappa^2/\mu^2)}\frac{I^2}{A} \qquad (44)$$

$$= +\frac{b}{2}\frac{I^2}{A}.$$

The magnitude of $F(R)$ is small for heavy nuclei and changes slowly with changes in $R = r_0 A^{1/3}$; for this reason the magnitude of b, which determines λ, and E_I, is practically independent of A.

From Eqs. (41) and (44) it follows that :

$$\lambda = -(\partial E_I/\partial I)_A = -(\partial E/\partial I)_A\,, \qquad (45)$$

which is in accord with the physical meaning of the Lagrange multiplier λ which corresponds to condition (21b) of the variational problem. For the case in which ρ signifies charge density, $I = A - 2Z$, and Eq. (44) assumes the form

$$E_I = 2b\left(\frac{A}{2} - Z\right)^2 / A. \qquad (46)$$

We have obtained, in this manner, the "isotopic" term of a semi-empirical formula for the energy of a nucleus [Ref. 4, Eq. (1.8)] with coefficient $2b$, determined by Eq. (44). This coefficient can be presented in a more expanded form by taking into account the value of $K\rho_0$, as well as Eqs. (18) and (28) and the relationship between ρ_0 and r_0:

$$2b = \left(\frac{3}{\pi}\right)^{1/3}\frac{1}{r_0^2} \qquad (47)$$

$$\frac{\frac{\pi}{2}\frac{\hbar^2}{M} + \frac{g^2}{\varkappa}\frac{2}{x}\left[\alpha_{1,2}\,\mathrm{lh}\,(1+x^2) - (\alpha + \alpha_{1,2})\left(1 - \frac{\ln(1+x^2)}{x^2}\right)\right]}{[1-(\varkappa^2/\mu^2)]\,F(R) + (\varkappa^2/\mu^2)}.$$

Also, on the basis of Eq. (36a)

$$\varkappa^2/\mu^2 = \psi/[\psi - (\beta - \beta_{1,2})(g^2/\varkappa)\,x], \qquad (48)$$

where ψ is the numerator of the fraction in Eq. (47).

On the basis of these formulas it is possible to determine the numerical value of the coefficient $2b$ in Eq. (46) for different types of forces, consistent with the requirement $\rho_0 \approx$ const.

For the interaction which contains an operator of type 4, Table 2, with parameters r_0, g^2/κ, γ, we obtain from the previous section, $2b \approx 100$ MeV, which agrees in order of magnitude with the experimentally determined value of 77.3 MeV.

In the case when ρ signifies spin state density with saturation of charges, $I = 2S$ in Eqs. (43) - (45), and we obtain a positive term in the expression for the energy of the nucleus having the form:

$$E_S = 2bS^2/A \qquad (49)$$

(S - resultant spin of the nucleus). But nuclei with saturated charges (the number of protons equal to the number of neutrons) and unsaturated spin states must belong to the group "A - even, Z - odd", and specifically for them, the additional empirical term in the expression $\delta(A, Z)$ for the energy of a nucleus [Ref. 4, Eqs. (1.8) and (1.9)] is positive.

The coefficient $2b$ is computed, by use of the same formulas (47), (48), as in the case of an isotopic member, but with different values of the coefficients $\alpha_{1,2}$, β - $\beta_{1,2}$ in correspondence with Table 2 (case "b" instead of case "a"). The computed magnitude of $2b$ in this case, with the operator and constants used in the computation of $2b$ for isotopic term is approximately 80 MeV.

Although Eq. (49) differs in appearance from the corresponding empirical term ($\sim A^{-3/4}$), it is interesting to note that for the heavier, stable nuclei of the type studied (N^{14}), we have the theoretical value $E_s \approx 6$ MeV, and the empirical, ≈ 4.7 MeV. The same type of correlation is obtained also for nuclei of other types.

Taking into consideration the roughness of our approximations, particularly for the case of unsaturated spin states (in connection with the neglected tensor forces), the correspondence in sign and order of magnitude of the theoretical formulas and empirical formulas indicates that the effect of unsaturated spin states, as also the isotopic effect, is correctly indicated by the statistical theory.

In closing we wish to call attention to the fact that the results of the present work indicate the applicability to research on the properties of heavy nuclei of the statistical method, which was developed here on the basis of nuclear forces that retain their basic properties predicted by meson theory (the type of distance function, exchange and spin terms). By choosing the parameters expressing these forces, corresponding both with the basic properties of the deuteron and the phenomenon of scattering of slow nucleons by nucleons, as well as with the property of saturation found in complex nuclei, proper orders of magnitude and sign have been obtained for various terms in the expression of binding energy of heavy nuclei. Also, the correct relationship between the radius of action of nuclear forces to the radius of space available to one nucleon has been found, as well as the explanation for the behavior of certain types of complex nuclei based on the analysis of influence of exchange and spin terms.

The method used here can be generalized to include tensor and Coulomb forces, which would improve the accuracy of the results of computations[6,7] dealing with the quadrupole moments of the nuclei. It must also be pointed out that, in the work reported by Gombas[8] which appeared after the preparation of this article, the statistical method is analyzed for application to nuclei with saturated spins. This work differs from ours both in the types of nuclear interactions (purely exchange forces between proton and neutron, and also, as in Bethe and Bacher [9], Sec. 6, spin dependent forces between like particles) and also in the method of derivation of theory (by means of solving various problems and analysis of related questions). The interactions studied by Gombas[8] are equivalent in their effect to saturation of forces, and have in their expressions operators of type 3, Table 2, given in this paper.

Taking into account the difficulties of interpreting experimental data pertaining to scattering of fast nucleons by nucleons[10] on the basis of semi-empirical nuclear forces, it may be appropriate to call attention to the semi-empirical character of the modern nuclear statistical theory the further refinement of which will be apparently concerned with the development of the meson theory of nuclear forces.

Translated by B. S. Maximoff
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