what artificially from the point of view of Gel'fand and Iaglom's theory of particles of higher spin. It is proposed that the consideration of the problem in the infinite-dimensional case can be connected with the general theory of non-local fields, giving vectors in hilbert space[7].

In conclusion, I want to express my thanks to Professor D. D. Tvanenko for a series of remarks and numerous discussions of the results of this work.

Translated by A. S. Wightman
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[7] D. Ivanenko and A. Sokolov, Klassische Feld. Theorie, Berlin (1953)

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On Longitudinal Vibrations of Plasma, 1.

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The problem of propogation of longitudinal waves in a plasma under given boundary conditions is solved. A "dispersion" equation which takes into account motion of the ions is obtained.

1. INTRODUCTION

IN studying the processes discovered by Langmuir, of vibrations in a plasma which arise because of the Coulomb interaction of charged particles, one usually considers only the motion of the electrons. The method proposed by Vlasov[1] for investigating specific electrical properties of plasma--the self-consistent field method--basically is applied only to them. Because of their large mass, the ions are considered as a positive background having no effect on the vibration of the electrons.

It is true that recently the necessity for considering ionic vibrations has been pointed out in several papers[2-4] However, no detailed investigations like those for electrons[5,6] exist in the literature*.

Ions can play a significant role in the vibrational properties of a plasma. In fact, since the mean free paths of electrons and ions are comparable, the squares of their transitional velocities are inversely proportional to their masses M/m. Therefore, during the passage of electrons and ions through a region of varying potential (an oscillating electrical double layer, etc.) the amplitude of vibration of the ions may turn out to be comparable to that of the electrons or even to exceed it, if the time for traversal of the region is smaller than the period of the variable potential. This behavior is clear, since the time for traversal of the region by the ions is M/m greater than for the electrons, although the acceleration given to the ion is a factor m/M smaller than that of an electron.

The ions play a special role when we consider the peculiar auto-oscillation process in plasma which leads to the possibility for occurrence of practically undamped waves, despite the occurrence of collisions[3]. The presence of two streams of charged particles (electrons and ions) with different velocities results in the appearance of undamped oscillations if the losses of energy from the wave due to collisions are compensated by a transfer of energy from the directed motion to the wave via the Coulomb interaction of the two particle streams.

A rigorous solution of the problem of longitudinal plasma oscillations (not considering the ions or collisions) was given in a paper of Landau[8] for the case where the initial deviation of the distribu-

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[1] A. A. Vlasov, J. Exper. Theoret. Phys. USSR 8, 291 (1938); Scientific Reports, Moscow State University 75, Book II, part I (1945); Theory of Many Particles, State Publishing House, (1950)

[2] G. Ia. Miakishev, Dissertation, Moscow State University, (1952)

[3] M. E. Gertzenshtein, J. Exper. Theoret. Phys. USSR 23, 669 (1952)

[4] V. P. Silin, J. Exper. Theoret. Phys. USSR 23, 649 (1952)

[5] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851, 1864, (1949)

[6] Yu. L. Klimontovich, Dissertation, Moscow State University (1951); J. Exper. Theoret. Phys. USSR 21, 1284, 1292 (1951)

* We consider the paper of Bazarov [e.g., see I. P. Bazarov, J. Exper. Theoret. Phys. USSR 21, 711 (1951)], which investigates the effect of the ions on the propagation of longitudinal waves in a plasma, to be unsatisfactory . In particular, in deriving the initial dispersion equation which is the basis of the investigation, an algebraic error is made as a result of which electronic and ionic terms which are proportional to the square of the charge have different signs in Bazarov's equation.

[8] L. D. Landau, J. Exper. Theoret. Phys. USSR 16, 574 (1946)

tion function from equilibrium is given (initial value problem). Up to the present time there has been no formulation and rigorous solution of the problem of plasma oscillation under assigned boundary conditions. This is precisely the problem which occurs when we consider plasma oscillations (e.g. in discharge tubes) which have stationary character.

The purpose of the present paper is to investigate stationary plasma oscillations caused by Coulomb forces, for given conditions at the boundary, and to analyze the part which ions play in the propagation of longitudinal waves in a plasma.

Keeping in mind that there are observed in plasmas sinusoidal oscillations with amplitudes not related to the magnitude of the ionization potential, an attempt is made in paper II (*v.* the following paper) to identify the longitudinal density waves considered here with the experimentally observed striations, and to compare the theory with experiment.

2. FORMULATION OF THE PROBLEM

The basis of all our later considerations is the set of kinetic equations for electrons and ions which takes into account the interaction of the charged particles by using the method of the self-consistent field in the form proposed by Vlasov [1]. Elastic collisions of charged particles with neutrals are included by means of a Boltzmann integral term.

The initial system of equations has the following form:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}\operatorname{grad}(\varphi + \varphi_{\text{ext}})\frac{\partial f_1}{\partial \mathbf{v}} = \iint (f_1' f_a' - f_1 f_a)\, V_1\, d\sigma_1 (d\mathbf{v}_a), \tag{1}$$

$$\frac{\partial f_2}{\partial t} + \mathbf{v}\frac{\partial f_2}{\partial \mathbf{r}} + \frac{e}{M}\operatorname{grad}(\varphi + \varphi_{\text{ext}})\frac{\partial f_2}{\partial \mathbf{v}} = \iint (f_2' f_a' - f_2 f_a)\, V_2\, d\sigma_2 (d\mathbf{v}_a), \tag{2}$$

$$\Delta\varphi = -4\pi e\left(\int_{-\infty}^{+\infty} f_1 (d\mathbf{v}) - \int_{-\infty}^{+\infty} f_2 (d\mathbf{v})\right), \tag{3}$$

where $f_1 = f_1(\mathbf{r}, \mathbf{v}, t)$, $f_2 = f_2(\mathbf{r}, \mathbf{v}, t)$ and $f_a = f_a(\mathbf{r}, \mathbf{v}, t)$ are the distribution functions for electrons, ions and atoms in the spatial coordinate $\mathbf{r}$ and velocity $\mathbf{v}$, depending on the time t; V_1 is the relative velocity of electron and atom, V_2 the relative velocity of ion and atom; $d\sigma_1$ and $d\sigma_2$ are the differential cross sections for collision of electrons and ions with atoms; e is the algebraic value of the electron charge; m and M are the electron and ion masses; ϕ_{ext} is the potential of the external electric field in the interior of the plasma, which is responsible for the drift of electrons and ions; ϕ is the total potential of the self-consistent field.

An arbitrary disturbance in some portion of the plasma will be taken care of by assigning a corresponding boundary condition (concerning this, more later). In the absence of perturbations, the system of equations (1) to (3) determines a distribution of electrons f_{10}, and of ions f_{20}, which are time-independent. We shall consider a quasineutral plasma for which, in the absence of perturbations, the mean densities of ions and electrons are equal, i.e.

$$\int_{-\infty}^{+\infty} f_{10} (d\mathbf{v}) = \int_{-\infty}^{+\infty} f_{20} (d\mathbf{v}).$$

For the case of a uniform distribution of particles in space in the absence of perturbing forces, the system of equations (1) to (3) splits up into two independent equations:

$$-\frac{e}{m}\operatorname{grad}\varphi_{\text{ext}}\frac{\partial f_{10}}{\partial \mathbf{v}} = \iint (f_{10}' f_a' - f_{10} f_a)\, V_1\, d\sigma_1 (d\mathbf{v}_a), \tag{4}$$

$$\frac{e}{M}\operatorname{grad}\varphi_{\text{ext}}\frac{\partial f_{20}}{\partial \mathbf{v}} = \iint (f_{20}' f_a' - f_{20} f_a)\, V_2\, d\sigma_2 (d\mathbf{v}_a). \tag{5}$$

Equations like Eq. (4) have been solved by many authors. For weak constant currents, under the assumption that the cross section for elastic collision of electrons with gas atoms is inversely proportional to their velocity, the distribution f_{10} as shown in papers (6, 9) is Maxwellian around the drift speed of the electrons. Under the condition that the electrons, in moving through a free path, receive energy far in excess of the thermal energy of the gas atoms, the distribution has the form:

$$f_{10} = N\left(\frac{m}{2\pi\Theta_1}\right)^{3/2} \exp\left\{-\frac{m(\mathbf{v} - \mathbf{v}_{01})^2}{2\Theta_1}\right\}, \tag{6}$$

where N is the electron concentration, $\Theta_1 = kT_1$, k is the Boltzmann constant, T_1 is the absolute temperature of the electrons, $\mathbf{v}_{01}$ is the mean velocity of their directed motion.

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[7] I. P. Bazarov, J. Exper. Theoret. Phys. USSR 21, 711 (1951)

[9] Taro Kihara, Revs. Mod. Phys. 24, 45 (1952)

The applicability of such a distribution to the case of high currents ($|\mathbf{v}_{01}| \gtrsim \mathbf{v}_{te} = 2\Theta_1/m$) has not been established theoretically.

The distribution function f_{20} of the ions is much more difficult to find than the distribution function for the electrons. Investigations in this direction have been carried out either for very special cases or under crude physical assumptions (Kihara[9], Fok[10] and others). Still, the use of a Maxwell distribution for the ions does not lead to any clear discrepancy with experiment, and is used by many authors (for example, Vlasov[1] and Klimontovich[6]). In our calculations, we shall also use as a zeroth approximation (no perturbation) a Maxwell distribution for the ions:

$$f_{20} = N\left(\frac{M}{2\pi\Theta_2}\right)^{3/2} \exp\left\{-\frac{M(\mathbf{v}-\mathbf{v}_{02})^2}{2\Theta_2}\right\}, \quad (7)$$

where $\Theta_2 = kT_2$, T_2 is the absolute temperature of the ions, $\mathbf{v}_{02}$ is the drift velocity of the ions.

We carry through a linearization of the initial equations (1)-(3), assuming that the redistribution of the plasma particles due to a given perturbation in some part of the plasma is small. Thus, we set:

$$f_1 = f_{10} + f_{11}, \quad f_2 = f_{20} + f_{21}, \quad (8)$$

noting that $f_{11} \ll f_{10}$, $f_{21} \ll f_{20}$, $\phi \ll \phi_{ext}$. Then we obtain for f_{11} and f_{21} (in the case of small currents, determined by the potential ϕ_{ext}, the following system of equations:

$$\frac{\partial f_{11}}{\partial t} + \mathbf{v}\frac{\partial f_{11}}{\partial \mathbf{r}} - \frac{e}{m}\operatorname{grad}\varphi\frac{\partial f_{10}}{\partial \mathbf{v}} = -\frac{1}{\tau_1}f_{11}, \quad (9)$$

$$\frac{\partial f_{21}}{\partial t} + \mathbf{v}\frac{\partial f_{21}}{\partial \mathbf{r}} + \frac{e}{M}\operatorname{grad}\varphi\frac{\partial f_{20}}{\partial \mathbf{v}} = -\frac{1}{\tau_2}f_{21}, \quad (10)$$

$$\Delta\varphi = -4\pi e\left(\int_{-\infty}^{+\infty} f_{11}(d\mathbf{v}) - \int_{-\infty}^{+\infty} f_{21}(d\mathbf{v})\right). \quad (11)$$

The right side of Eq. (9) is the usual approximation of the Boltzmann term for the electrons, which has been used in many papers concerning plasmas ($1/\tau_1$ is the electron-atom collision frequency). In just this same way the right side of Eq. (10) is the first approximation to the Boltzmann term for the ions in the case of small perturbations (τ_2 is the mean free time for the ions).

Over the plane $x = 0$, we preassign a perturbation which is periodic in time with frequency ω (it is sufficient to consider a periodic perturbation, since, by virtue of the linearity of the system of equations, the solution in the case of an arbitrary perturbation can be represented in the form of a Fourier series or integral with known harmonics). Then the solution of the system of Eqs. (9)-(11) describes the process of propagation of the given disturbance into the plasma. Since we are considering a plane problem,

$$f_{11} = f_{11}(x, \mathbf{v}, t),$$
$$f_{21} = f_{21}(x, \mathbf{v}, t) \text{ and } \phi = \phi(x, t).$$

For convenience we integrate the system of equations over the y and z components of the velocity. Once more designating the functions $\iint_{-\infty}^{+\infty} f_{11}\, d\eta\, d\zeta$ and $\iint_{-\infty}^{+\infty} f_{21}\, d\eta\, d\zeta$ thus obtained by f_{11} and f_{21}, we obtain

$$\frac{\partial f_{11}}{\partial t} + \xi\frac{\partial f_{11}}{\partial x} - \frac{eN}{m}\frac{\partial\varphi}{\partial x}\frac{\partial F_{1,0}}{\partial\xi} = -\frac{1}{\tau_1}f_{11}, \quad (12)$$

$$\frac{\partial f_{21}}{\partial t} + \xi\frac{\partial f_{21}}{\partial x} + \frac{eN}{M}\frac{\partial\varphi}{\partial x}\frac{\partial F_{2,0}}{\partial\xi} = -\frac{1}{\tau_2}f_{21}, \quad (13)$$

$$\Delta\varphi = -4\pi e\left(\int_{-\infty}^{+\infty} f_{11}\, d\xi - \int_{-\infty}^{+\infty} f_{21}\, d\xi\right), \quad (14)$$

where

$$F_{1,0} = \left(\frac{m}{2\pi\Theta_1}\right)^{1/2} \exp\left\{-\frac{m(\xi-\xi_{01})^2}{2\Theta_1}\right\}; \quad (15)$$

$$F_{2,0} = \left(\frac{M}{2\pi\Theta_2}\right)^{1/2} \exp\left\{-\frac{M(\xi-\xi_{02})^2}{2\Theta_2}\right\}.$$

The problem we have formulated--the propagation in a plasma of longitudinal waves arising from a perturbation at the boundary $x = 0$ which varies periodically with frequency ω (a boundary value problem) which is analogous to the problem of Landau[8] in which he investigated the time behavior of a perturbation in a plasma (initial value problem). Therefore, generally speaking, we can also apply to our problem the method of solution of Landau, which makes use of one-sided Laplace transformations applied to the initial equations (the problem for the half-space $[x = 0, x = \infty]$). In our case this method requires that at $x = 0$ the functions

$$f_{11}(0, \xi, t), \quad f_{21}(0, \xi, t),$$
$$\phi(0, t), \quad (\partial\phi/\partial x)(0, t).$$

shall be given.

However, in specifying our boundary value

[10] V. A. Fok, J. Exper. Theoret. Phys. USSR 18, 1049 (1948)

[11] E. Hopf, Mathematical problems of radiative equilibrium, Cambr. Tracts 31 (1933)

problem* we can assign arbitraily only the magnitude of the current of particles entering the medium through its boundary, so that we can assign the functions listed above only for positive values of ξ (we take the + ξ axis along + x). The form of these functions for negative values of ξ is already determined by their form for $\xi > 0$, since the current of particles emerging from the medium across its boundary is automatically regulated by processes going on in its interior (collisions, Coulomb interactions, etc.). In our case the decisive role in this respect is played by the elastic collisions of the charged particles with neutrals; this is made clear by the fact that the solution of our linearized system of equations in which we neglect collisions ($1/\tau_1 = 1/\tau_2 = 0$) and use the method of Landau, does not lead to any auxiliary conditions of consistency which must be satisfied by the functions $f_{11}(0, \xi, t)$, $f_{21}(0, \xi, t)$. At the same time, if we include collisions, this same method leads to an auxiliary condition for consistency, which must be satisfied by the functions $f_{11}(0, \xi, t)$ and $f_{21}(0, \xi, t)$ because of the dependence of the form of these functions for $\xi < 0$ on their values for $\xi > 0$. It is therefore natural to choose a method of solution of the problem which goes with the direct assignment of the distribution function for the particles only for $\xi > 0$. Such a method of solution was suggested to us by N. N. Bogoliubov, and is carried through in the the following section.

First we simplify the equation system (12)-(14) by making use of the fact that a perturbation on the boundary which varies periodically with frequency ω, results in propagation in the plasma of a perturbation which varies in time with this same frequency. Setting

$$f_{11}(x, \xi, t) = \psi_{11}(x, \xi) e^{-i\omega t},\quad f_{21}(x, \xi, t) = \psi_{21}(x, \xi) e^{-i\omega t},\quad \varphi(x, t) = \varphi_1(x) e^{-i\omega t}, \tag{16}$$

we obtain for the functions ψ_{11}, ψ_{21}, ϕ_1 the following system of equations:

$$-i\omega\psi_{11} + \xi\frac{\partial\psi_{11}}{\partial x} - \frac{eN}{m}\frac{\partial\varphi_1}{\partial x}\frac{\partial F_{1,0}}{\partial\xi} = -\frac{1}{\tau_1}\psi_{11}, \tag{17}$$

$$-i\omega\psi_{21} + \xi\frac{\partial\psi_{21}}{\partial x} + \frac{eN}{M}\frac{\partial\varphi_1}{\partial x}\frac{\partial F_{2,0}}{\partial\xi} = -\frac{1}{\tau_{21}}\psi_{21}, \tag{18}$$

$$\Delta\varphi_1 = -4\pi e\left(\int_{-\infty}^{+\infty}\psi_{11}\,d\xi - \int_{-\infty}^{+\infty}\psi_{21}\,d\xi\right). \tag{19}$$

*The specification of the boundary value problem was pointed out to us by A. N. Tikhonov and N. N. Bogoliubov.

3. SOLUTION OF THE PROBLEM

We shall solve the system of equations (17)-(19), subjecting the functions ψ_{11} and ψ_{21} to the following boundary conditions:

$$\psi_{11}(0, \xi) = f_1(\xi), \quad \xi > 0; \tag{20}$$

$$\psi_{21}(0, \xi) = f_2(\xi), \quad \xi > 0; \tag{21}$$

$$\psi_{11}(\infty, \xi) = 0, \tag{22}$$

$$\psi_{21}(\infty, \xi) = 0. \tag{23}$$

The last two conditions correspond to absorption at infinity. We could also consider similarly the case of a totally reflecting wall at one of the boundaries, or other cases.

We also want to point out the close connection between the formulation of our present problem of propagation of longitudinal waves in a plasma and that of the well known problem of Milne[11-13] concerning the scattering and absorption of light in the atmosphere.

A solution of Eqs. (17) and (18) for the functions $\xi\psi_{11}(x, \xi)$, $\xi\psi_{21}(x, \xi)$ satisfying the boundary conditions (20)-(23), can be given in the form

$$\xi\psi_{11}(x, \xi) = \xi f_1(\xi)\exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\} + \frac{eN}{m}\int_0^x \exp\left\{-\frac{x-x'}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\} \times \frac{\partial\varphi_1(x')}{\partial x'}\frac{\partial F_{1,0}(\xi)}{\partial\xi}dx', \quad \xi > 0, \tag{24}$$

$$\xi\psi_{11}(x, \xi) = -\frac{eN}{m}\int_x^\infty \exp\left\{-\frac{x-x'}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\}$$

[12] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (1948)

[13] V. A. Fok, Matem. Sbornik 14, 3 (1944)

$$\times \frac{\partial \varphi_1(x')}{\partial x'} \frac{\partial F_{1,0}(\xi)}{\partial \xi} dx', \quad \xi < 0;$$

$$\xi\psi_{21}(x, \xi) = \xi f_2(\xi) \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\}$$

$$-\frac{eN}{M}\int_0^x \exp\left\{-\frac{x-x'}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\}$$

$$\times \frac{\partial \varphi_1(x')}{\partial x'} \frac{\partial F_{2,0}(\xi)}{\partial \xi} dx', \quad \xi > 0,$$

$$\xi\psi_{21}(x, \xi) = \frac{eN}{M}\int_x^\infty \exp\left\{-\frac{x-x'}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\} \tag{25}$$

$$\times \frac{\partial \varphi_1(x')}{\partial x'} \frac{\partial F_{2,0}(\xi)}{\partial \xi} dx', \quad \xi < 0.$$

We introduce the following notation:

$$\int_0^\infty \xi f_1(\xi) \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\} d\xi = u_1(x); \tag{26}$$

$$G_1(x) = \int_0^\infty \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\} \tag{27}$$

$$\times \frac{\partial F_{1,0}(\xi)}{\partial \xi} d\xi, \quad x > 0,$$

$$G_1(x) = -\int_{-\infty}^0 \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1} - i\omega\right)\right\}$$

$$\times \frac{\partial F_{1,0}(\xi)}{\partial \xi} d\xi, \quad x < 0;$$

$$\int_0^\infty \xi f_2(\xi) \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\} d\xi = u_2(x), \tag{28}$$

$$G_2(x) = \int_0^\infty \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\} \tag{29}$$

$$\frac{\partial F_{2,0}(\xi)}{\partial \xi} d\xi, \quad x > 0,$$

$$G_2(x) = \int_{-\infty}^0 \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2} - i\omega\right)\right\}$$

$$\times \frac{\partial F_{2,0}(\xi)}{\partial \xi} d\xi, \quad x < 0.$$

Then

$$\int_{-\infty}^{+\infty} \xi\psi_{11}(x, \xi) d\xi = u_1(x) \tag{30}$$

$$+\frac{eN}{m}\int_0^\infty G_1(x - x') \frac{\partial \varphi_1(x')}{\partial x'} dx';$$

$$\int_{-\infty}^{+\infty} \xi\psi_{21}(x, \xi) d\xi = u_2(x) \tag{31}$$

$$-\frac{eN}{M}\int_0^\infty G_2(x - x') \frac{\partial \varphi_1(x')}{\partial x'} dx'.$$

In addition we make use of the equation of continuity associated with the system of equations (17)-(19):

$$\frac{\partial j_{11}(x)}{\partial x} \frac{1}{i\omega - (1/\tau_1)} + \frac{\partial j_{21}(x)}{\partial x} \frac{1}{i\omega - (1/\tau_2)} = \rho(x), \tag{32}$$

where

$$j_{11}(x) = e\int_{-\infty}^{+\infty} \xi\psi_{11}(x, \xi) d\xi, \tag{33}$$

$$j_{21}(x) = -e\int_{-\infty}^{+\infty} \xi\psi_{21}(x, \xi) d\xi,$$

$$\rho(x) = e\int_{-\infty}^{+\infty} \psi_{11}(x, \xi) d\xi - e\int_{-\infty}^{+\infty} \psi_{21}(x, \xi) d\xi.$$

Equations (32) and (19) lead to the relation

$$\frac{j_{11}(x)}{i\omega - (1/\tau_1)} + \frac{j_{21}(x)}{i\omega - (1/\tau_2)} = -\frac{1}{4\pi} \frac{\partial \varphi_1(x)}{\partial x}. \tag{34}$$

Setting

$$\frac{u_1(x)}{(1/\tau_1) - i\omega} - \frac{u_2(x)}{(1/\tau_2) - i\omega} = U(x), \tag{35}$$

we obtain from the relations (30), (31), and (34)

$$\frac{\partial \varphi_1(x)}{\partial x} = 4\pi e U(x) \tag{36}$$

$$+\frac{4\pi Ne^2}{m}\int_0^\infty G_1(x-x')\frac{\partial\varphi_1(x')}{\partial x'}\frac{dx'}{(1/\tau_1)-i\omega}$$

$$+\frac{4\pi Ne^2}{M}\int_0^\infty G_2(x-x')$$

$$\times\frac{\partial\varphi_1(x')}{\partial x'}\frac{dx'}{(1/\tau_2)-i\omega}.$$

We define

$$\partial\varphi_1(x)/\partial x=f(x), \tag{37}$$

$$\frac{4\pi Ne^2}{m}\frac{G_1(x)}{(1/\tau_1)-i\omega}+\frac{4\pi Ne^2}{M}\frac{G_2(x)}{(1/\tau_2)-i\omega}=k(x), \tag{38}$$

$$4\pi eU(x)=g(x). \tag{39}$$

From (36), we obtain for $f(x)$ an integral equation of the form:

$$f(x)=g(x)+\int_0^\infty k(x-x')f(x')\,dx' \tag{40}$$

with the kernel

$$k(x)=\omega_{10}^2\int_0^\infty \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1}-i\omega\right)\right\} \tag{41}$$

$$\times\frac{\partial F_{1,0}(\xi)}{\partial\xi}\frac{d\xi}{(1/\tau_1)-i\omega}$$

$$+\omega_{20}^2\int_0^\infty \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2}-i\omega\right)\right\}$$

$$\times\frac{\partial F_{2,0}(\xi)}{\partial\xi}\frac{d\xi}{(1/\tau_2)-i\omega},\quad x>0,$$

$$k(x)=-\omega_{10}^2\int_{-\infty}^0 \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1}-i\omega\right)\right\}$$

$$\times\frac{\partial F_{1,0}(\xi)}{\partial\xi}\frac{d\xi}{(1/\tau_1)-i\omega}$$

$$-\omega_{20}^2\int_{-\infty}^0 \exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2}-i\omega\right)\right\}$$

$$\times\frac{\partial F_{2,0}(\xi)}{\partial\xi}\frac{d\xi}{(1/\tau_2)-i\omega},\quad x<0,$$

where $\omega_{10}=\sqrt{4\pi Ne^2/m}$ and $\omega_{20}=\sqrt{4\pi Ne^2/M}$ are the Langmuir oscillation frequencies of the electrons and ions, respectively. The free term has the form

$$g(x)=4\pi e\int_0^\infty \xi f_1(\xi)\exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1}-i\omega\right)\right\} \tag{42}$$

$$\times\frac{d\xi}{(1/\tau_1)-i\omega}$$

$$-4\pi e\int_0^\infty \xi f_2(\xi)\exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_2}-i\omega\right)\right\}$$

$$\times\frac{d\xi}{(1/\tau_2)-i\omega}.$$

Equations like (40) come up in many problems of mathematical physics [13]. For example, the previously mentioned Milne problem concerning the scattering and absorption of light in the atmosphere coincides with ours in its formulation and leads to an equation of this same type. An equation of this type was investigated in detail by Fok [13] who gives the method of solution, and proves the existence and uniqueness of a solution falling off at infinity ($x\to\infty$), under the following assumptions concerning the kernel $k(x)$ and the function $g(x)$*: the kernel itself and the functions $g(x)$, $k_1(x)=e^{c|x|}k(x)$, for some $c>0$, shall be absolutely integrable and have bounded variation in an infinite interval. If the equation $1-K(w)=0$ $\left(K(w)=\int_{-\infty}^{+\infty}k(x)e^{iwx}dx\right)$ has real roots in addition to com-

* Fok considered an equation with a symmetric kernel; but one can show that, without changing his argument essentially, the basic results remain valid also for an unsymmetric kernel.

plex roots, one must add to the previous conditions the requirement that these same conditions be satisfied by the function $(\ln g(x))\,x^{s-1}g(x)$ (where s is the maximum multiplicity of the root) and the orthogonality condition: $\int_0^\infty g(x)\,\gamma_r(x)\,dx = 0$ (where the $\gamma_r(x)$ are any linearly independent solutions of the homogeneous equation).

The asymptotic behavior of the solution is determined by the equation

$$1 - K(w) = 0 \tag{43}$$

or, defining

$$-iw = p, \quad \int_{-\infty}^{+\infty} e^{-px} k(x)\,dx = K(p), \tag{44}$$

by the equation

$$1 - K(p) = 0, \tag{45}$$

where $K(p)$ is analytic in the strip $-c < \mathrm{Re}\,p < c$: in fact for $x \to \infty$ the solution has the form

$$f(x) \sim \exp\{p_k x\}, \tag{46}$$

where p_k is that root of Eq. (45) with $\mathrm{Re}\,p < 0$ which is closest to the imaginary axis.

In order to apply Fok's method to the solution of our Eq. (40), we must first verify that the kernel $k(x)$ defined by (41) satisfies the required conditions.

Keeping in mind that particles with infinitely large velocities do not affect processes in the plasma, we shall, for convenience in applying the above mentioned method of solution, set

$$F_{1,0}(\xi) = F^{M}_{1,0}(\xi), \quad |\xi| < a, \tag{47}$$

$$F_{1,0}(\xi) = AF^{M}_{1,0}(\xi)\,\frac{\exp\{-N_{01}(\xi)\}}{\exp\{-m(\xi-\xi_{01})^2/2\Theta_1\}}, \quad |\xi| > a;$$

$$F_{2,0}(\xi) = F^{M}_{2,0}(\xi), \quad |\xi| < b, \tag{48}$$

$$F_{2,0}(\xi) = BF^{M}_{2,0}(\xi)\,\frac{\exp\{-N_{02}(\xi)\}}{\exp\{-M(\xi-\xi_{02})^2/2\Theta_2\}},$$

where

$$N_{01}(\xi) = \exp\{|\xi-\xi_{01}|^{\alpha}(m/2\Theta_1)^{\alpha/2}\}$$

$$N_{02}(\xi) = \exp\{|\xi-\xi_{02}|^{\beta}(M/2\Theta_2)^{\beta/2}\},$$

$F^{m}_{1,0}(\xi)$, $F^{m}_{2,0}(\xi)$ are Maxwellian velocity distributions for the electrons and ions, as given by Eq. (15); a, b, α, β are arbitrary constants satisfying the conditions: $a \gg 2|\xi_{01}|$, $b \gg 2|\xi_{02}$, $0 < \alpha < 1$, $0 < \beta < 1$; the constants A and B are determined from the condition of continuity of the functions $F_{1,0}(\xi)$ and $F_{2,0}(\xi)$:

$$A = \frac{\exp\{-m(a-\xi_{01})^2/2\Theta_1\}}{\exp\{-N_{01}(a)\}}, \tag{49}$$

$$B = \frac{\exp\{-M(b-\xi_{02})^2/2\Theta_2\}}{\exp\{-N_{02}(b)\}}.$$

It is not difficult to see that $k(x)$ is a continuous function of x, since the integrals defining $k(x)$ converge uniformly in x and the functions in the integrands are continuous. Consequently we need only investigate the behavior of $k(x)$ at infinity. One can obtain an upper bound for the modulus of $k(x)$:

$$|k(x)| < C|x|e^{-c|x|}, \tag{50}$$

where

$$c = \min\{1/a\tau_1,\ 1/b\tau_2\}, \tag{51}$$

$$C = \max\Big\{2(1+A)\Big(\frac{m}{2\pi\Theta_1}\Big)^{1/2} \times \frac{\omega_{10}^2}{|(1/\tau_1) - i\omega|\,\tau_1(a-|\xi_{01}|)} \times 2(1+B)\Big(\frac{M}{2\pi\Theta_2}\Big)^{1/2} \times \frac{\omega_{20}^2}{|(1/\tau_2) - i\omega|\,\tau_2(b-|\xi_{02}|)}\Big\} \tag{52}$$

The estimate obtained for $k(x)$ shows the following:

1) $k(x)$ is an absolutely integrable function in the infinite interval $(-\infty, +\infty)$.

2) $e^{c'|x|}k(x)$ is also an absolutely integrable function in the infinite interval $(-\infty, +\infty)$ for $0 < c' < c$.

Differentiating $k(x)$ under the integral sign, one can show in just this same way that $k'(x)$ is an absolutely integrable function in the infinite interval. At infinity, $k'(x)$ decreases no more slowly than an exponential with exponent $cx/2$. It is obvious that $\{e^{c''|x|}k(x)\}'$ is also absolutely integrable in the infinite interval for $c'' < c/2$. The condition of absolute integrability of the derivative is sufficient to make the function have bounded variation in the infinite interval. Consequently, the functions $k(x)$ and $e^{c''|x|}k(x)$ have bounded variation in that interval.

It is not hard to see that the function $g(x)$, defined in terms of the arbitrary functions $f_1(\xi)$, $f_2(\xi)$ by (42), also satisfies the requirements imposed, if $f_1(\xi)$ and $f_2(\xi)$ decrease sufficiently rapidly at infinity.

The preceeding discussion allows us to conclude that Fok's method applies to the solution of Eq. (40). Therefore we can assert that there exists a unique solution of (40), falling off at infinity, and that its asymptotic behavior is given by the "dispersion" Eq. (45).

We transform this last equation to a somewhat different form. For this purpose, we calculate $\int_{-\infty}^{+\infty} k(x)e^{-px}dx$, substituting for $k(x)$ its expression in (41). We carry out the computation for the electronic part of the kernel, $k_e(x)$.

$$\int_{-\infty}^{+\infty} k_e(x)e^{-px}dx = \omega_{10}^2\int_0^\infty\left\{\int_0^\infty \frac{\partial F_{1,0}}{\partial\xi} \times \frac{\exp\{-(x/\xi)[(1/\tau_1)-i\omega]\}}{(1/\tau_1)-i\omega}d\xi\right\}e^{-px}dx - \omega_{10}^2\int_{-\infty}^0\left\{\int_{-\infty}^0 \frac{\partial F_{1,0}}{\partial\xi} \times \frac{\exp\{-(x/\xi)[(1/\tau_1)-i\omega]\}}{(1/\tau_1)-i\omega}d\xi\right\}e^{-px}dx. \tag{53}$$

For values of p with $\mathrm{Re}\,p > 0$, we can, in computing the first integral, reverse the order of integration; then

$$\omega_{10}^2\int_0^\infty\left\{\int_0^\infty \frac{\partial F_{1,0}}{\partial\xi}\frac{\exp\{-(x/\xi)[(1/\tau_1)-i\omega]\}}{(1/\tau_1)-i\omega}d\xi\right\} \times e^{-px}dx = -\frac{\omega_{10}^2}{p^2}\int_0^\infty \frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{\xi-(i\omega/p)+(1/\tau_1 p)} + \frac{\omega_{10}^2}{p}\int_0^\infty \frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{(1/\tau_1)-i\omega}. \tag{54}$$

We obtain a similar expression for the second integral, calculated for values of p with $\mathrm{Re}\,p < 0$:

$$-\omega_{10}^2\int_{-\infty}^0\left\{\int_{-\infty}^0 \frac{\partial F_{1,0}}{\partial\xi}\exp\left\{-\frac{x}{\xi}\left(\frac{1}{\tau_1}-i\omega\right)\right\} \times \frac{d\xi}{(1/\tau_1)-i\omega}\right\}e^{-px}dx = -\frac{\omega_{10}^2}{p^2}\int_{-\infty}^0 \frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{\xi-(i\omega/p)+(1/\tau_1 p)} + \frac{\omega_{10}^2}{p}\int_{-\infty}^0 \frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{(1/\tau_1)-i\omega} \tag{55}$$

But, since $\int_{-\infty}^{+\infty} k_e(x)e^{-px}dx$ is an analytic function of the complex variable p (cf. page 18) in the strip $-c'' < \mathrm{Re}\,p < c''$, we obtain the expression for $\int_{-\infty}^{+\infty} k_e(x)e^{-px}dx$ by analytic continuation of the sum of the integrals (54), (55) over the strip $-c'' < \mathrm{Re}\,p < c''$. The ionic part of $K(p)$ is calculated similarly. Finally we have

$$\int_{-\infty}^{+\infty} k(x)e^{-px}dx = -\frac{\omega_{10}^2}{p^2} \times \int_{C_1} \frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{\xi-(i\omega/p)+(1/\tau_1 p)} \tag{56}$$

$$-\frac{\omega_{20}^2}{p^2}\int_{C_2}\frac{\partial F_{2,0}}{\partial\xi}\frac{d\xi}{\xi-(i\omega/p)+(1/\tau_2 p)},$$

where the contours C_1 and C_2 go along the real axis from $\xi=-\infty$ to $\xi=+\infty$ circling the points $\xi_p=(i\omega/p)-(1/\tau_1 p)$ (for the contour C_1) and $\xi_p=(i\omega/p)-(1/\tau_2 p)$ (for contour C_2) from below.

Now the "dispersion" equation defining the asymptotic form of the solution for the longitudinal electric field produced in a plasma by a periodic perturbation at the boundary can be written in the form

$$1+\frac{\omega_{10}^2}{k^2}\int_{C_1}\frac{\partial F_{1,0}}{\partial\xi}\frac{d\xi}{(\omega/k)+(i/\tau_1 k)-\xi}+\frac{\omega_{20}^2}{k^2}\int_{C_2}\frac{\partial F_{2,0}}{\partial\xi}\frac{d\xi}{(\omega/k)+(i/\tau_2 k)-\xi}=0, \quad (57)$$

where $k=ip$, and the contour C_1 bypasses the point $\omega/k+i/\tau_1 k$, and the contour C_2 similarly bypasses the point $\omega/k+i/\tau_2 k$ from below.

At sufficiently large distances from the source of the perturbation, the distribution of field (and particles) will be approximated to sufficient accuracy by an expression of the form $A\exp\{-i(\omega t-kx)\}$ (where k is the root of Eq.(57) which is closest to the imaginary axis). The quantity $2\pi/\mathrm{Re}\,k$ represents the spatial period of the propagating disturbance, while $\mathrm{Im}\,k$ is its logarithmic decrement*.

The last equation differs from the one obtained by Landau [8] for the dispersion of longitudinal waves in a plasma due to a given perturbation at the initial time, by terms which take into account the ions and the collisions of charged particles with neutrals. In addition, in Eq. (57) k is complex and ω is a real number ($\omega>0$), whereas in Landau's equation ω was complex and k was a real number.

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Translated by M. Hamermesh
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* The "dispersion" equation (57) also gives the criterion for occurrence of antidamped solutions, although a rigorous solution of the problem for this case (i.e., a determination of the amplitude) cannot be given within the realm of the linear theory.