The Theory of Nuclear Reactions with Production of Slow Particles*

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Energy and angular distribution of slow nucleons which result from nuclear reactions are obtained. An estimate is given of the probability of production of nucleons in a bound state.

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1. IF as a result of nuclear reactions particles with low kinetic energy are produced, the mutual interaction of these particles may substantially influence their distribution in energy. The case will be considered in which two or more nucleons are obtained as reaction products, each with kinetic energy which is small compared to the interaction energy.

In this paper we obtain the distribution in energy of the nucleons which result from collisions of a nucleon with a deuteron and a deuteron with nuclei. In order to obtain the distribution in energy of the slow nucleons which are produced, it is sufficient to make use of quite general properties of their ψ-functions, and hence it is also possible to make some inferences concerning the distribution in energy for more complicated nuclear reactions. Below we give an estimate of the ratio of the cross-section for deuteron formation (or the dineutron, if it exists) to the cross-section for production of free nucleons. We also obtain the angular correlation between exit directions of the two nucleons which results from their interaction.

2. Let a nucleon be incident on a deuteron, and let the velocity of the nucleon be much greater than the velocities of the neutron and proton in the deuteron. In order to obtain the cross-section for this process we shall use a method analogous to the one used in molecular theory, where one makes use of the smallness of the velocities of the nuclei as compared with the velocities of the electrons. In analogy with the molecular case, it is necessary, in order to find zero order functions, to solve first the problem of the scattering of the incident particle by the fixed nucleons of the deuteron. The zero order functions will have the form

$$\Psi = \psi(\mathbf{r}_3;\ \mathbf{r}_1, \mathbf{r}_2)\,\varphi(\mathbf{r}_1, \mathbf{r}_2),$$

where ψ is the function of the incident particle in the field of fixed particles of the deuteron, ϕ is a function of the slow nucleons.

If the scattering function of the incident particle with neutron and proton as scatterers is known, then neglecting the influence of the neutron on the function describing the scattering on the proton, and vice versa (this is legitimate for high energies of the incident particle, when $\sqrt{\sigma} \ll \mathbf{r}_d$, where σ is the scattering cross-section and $\mathbf{r}_d$ is the radius of the deuteron), we obtain the asymptotic form of the function ψ

$$(\psi)_{r_3\to\infty} \sim e^{i\mathbf{k}_0\mathbf{r}_3} + f_1\,(e^{ik_0 r_1}/r_3)\,e^{ik|\mathbf{r}_3-\mathbf{r}_1|}$$

$$+ f_2\,(e^{i\mathbf{k}_0\mathbf{r}_2}/r_3)\,e^{ik|\mathbf{r}_3-\mathbf{r}_2}$$

$$= e^{i\mathbf{k}_0\mathbf{r}_3} + (f_1 e^{i\mathbf{q}\mathbf{r}_1} + f_2 e^{i\mathbf{q}\mathbf{r}_2})\,(e^{ikr_3}/r_3)$$

where $\mathbf{q} = \mathbf{k}_0 - \mathbf{k}$, $\mathbf{k} = k\mathbf{r}_3/r$, and f_3 and f_2 are scattering amplitudes of the incident particle on the nucleons 1 and 2.

The wave function of the system of all three-particles will be determined by the condition that when $\mathbf{k}_0\mathbf{r}_3 \to -\infty$, the wave function must have the form $e^{i\mathbf{k}_0\mathbf{r}_3}\phi_0(\mathbf{r}_1,\mathbf{r}_2)$ where $\phi_0(\mathbf{r}_1,\mathbf{r}_2)$ is the wave function of the deuteron. The asymptotic form of the wave function of the entire system in zero approximation is given by the expression

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Note added in proof: Since completion of this work several papers dealing with this problem have appeared [see, e.g., K. M. Watson, Phys. Rev. 88, 1163 (1952)].

$$\Psi_{r_3\to\infty} \sim e^{i\mathbf{k}_0\mathbf{r}_3}\varphi_0(\mathbf{r}_1,\mathbf{r}_2) + (f_1 e^{i\mathbf{q}\mathbf{r}_1} + f_2 e^{i\mathbf{q}\mathbf{r}_2})\varphi_0(\mathbf{r}_1,\mathbf{r}_2)(e^{ikr_3}/r_3).$$

Expanding the expression

$$(f_1 e^{i\mathbf{q}\mathbf{r}_1} + f_2 e^{i\mathbf{q}\mathbf{r}_2})\varphi_0(\mathbf{r}_1,\mathbf{r}_2)$$

in terms of the eigenfunctions of the two slow nucleons produced as a result of the collision, and squaring the expansion coefficients, we obtain the cross-section for scattering with production of nucleons in various states. Introducing coordinates $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$; $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, we obtain, after integration with respect to $\mathbf{R}$

$$d\sigma = \left|\int \sum \varphi_P^*(\mathbf{r})\chi_1\{f_1 e^{i\mathbf{q}\mathbf{r}/2} + f_2 e^{-i\mathbf{q}\mathbf{r}/2}\}\varphi_0(\mathbf{r})\chi_0 d\mathbf{r}\right|^2 d\omega_k\, d\mathbf{P}/(2\pi\hbar)^3, \tag{1}$$

where $d\omega_k$ is the element of solid angle containing the vector $\mathbf{k}$; χ_0 and χ_1 are functions of the charge and spin coordinates of the two nucleons. The functions $\phi_p(\mathbf{r})$ are normalized to unit volume, function ϕ_0 to unity. The summation is with respect to the spin and charge variables of the two nucleons; P is the momentum of particles 1 and 2 in their center of mass system.

The cross-section for production of two nucleons in a bound state has the form

$$d\sigma_0 = \left|\int \sum \varphi_1^*\chi_1\{f_1 e^{i\mathbf{q}\mathbf{r}/2} + f_2 e^{-i\mathbf{q}\mathbf{r}/2}\}\varphi_0\chi_0 d\mathbf{r}\right|^2 d\omega_k, \tag{2}$$

with functions ϕ_1 normalized to unity. In the case of collisions without exchange the function ϕ_1 coincides with ϕ_0. Scattering amplitudes f_1 and f_2 contain, in general, exchange terms in spin and charge variables. Hence during the scattering process changes may take place in the spin and charge states of the nucleons.

3. The integrals in the expressions (1) and (2) cannot be calculated for large q since in this case values of $r \sim 1/q$ are important while the functions ϕ_p and ϕ_0 are known only for $r > r_0$, where r_0 is the range of the interaction forces. For what follows only the dependence of Eq. (1) on the interaction energy of the nucleons is essential. This dependence may be easily found for the case that $\hbar q$ is large compared to the momentum of the particles in the deuteron and compared to the momentum P. In this case in the integrals of Eq. (1) the region of small $r (r \sim 1/q)$ is essential, and the probability of finding the nucleons with relative wave vector $\mathbf{P}$ which lies in the interval $d\mathbf{P}$, is

$$dw_p = C_1|\varphi_P(\mathbf{r}_1)|^2 d\mathbf{P};\quad r_1 \sim 1/q. \tag{3}$$

4. Let us consider the case of a non-exchange collision. Then one has in the final state (besides the fast nucleon) a slow neutron and a proton. As is easy to obtain from the theory of scattering of neutrons by protons*,

$$|\varphi_P(r_1)|^2 = \frac{A}{E+\varepsilon}\left\{1 + O\left(\frac{P^2 r_0^2}{\hbar^2}\right)\right\},\qquad P^2 \ll P_{1,2}^2,$$

where A does not depend on the energy, E is the energy of relative motion of the nucleons, $\epsilon = \epsilon_0 =$ 2.2 MeV for parallel spins and $\epsilon = \epsilon_1 =$ 0.07 MeV for antiparallel spins of neutron and proton. Substituting this expression into (3), it is easy to obtain

$$dw_P^{np} = C_{np}\left\{\frac{1}{E+\varepsilon_0} + \frac{a}{E+\varepsilon_1}\right\}d\mathbf{P}. \tag{4}$$

The quantity a determines the amount of admixture of the component depending on the spin in the non-exchange scattering amplitude. The angular distribution of the nucleons in their center-of-mass system is spherically symmetrical.

Formula (4) shows that the energy distribution of the nucleons has a maximum at energies of the order of ϵ.

5. When, as a result of the exchange interaction with the incident fast particle two slow neutrons are formed, it can be seen from (3) that they must be in an S-state (functions with non-zero orbital momentum are small for small distances r_1). Then, according to the exclusion principle, their spins are anti-parallel, and during the collision a change of spin must occur (in the deuteron the spins of the nucleons are parallel).

The distribution in relative energies is given by the expression

$$dw_P^{nn} = C_{nn}d\mathbf{P}/(E+\varepsilon). \tag{5}$$

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* If $r_1 < r_0$ the dependence of $\phi_p(r_1)^2$ on the energy is determined by the dependence on the energy of the expression $\phi_p(r_0)^2$ since when $r < r_0$ one can neglect the energy E compared to depth of potential well.

The quantity ϵ is in this case unknown.

6. In the case of an exchange collision with formation of two slow protons the function $\phi_p(r_1)$ may be obtained from the theory of proton-proton scattering[1,2]:

$$\varphi_P(r_1) = \frac{\hbar}{r_1}\sqrt{\frac{F(\eta)}{M}}\left\{F^2(\eta)E + \frac{\hbar^2}{M}\left[-\frac{1}{a} - \frac{h(\eta)}{R} + \gamma E\right]^2\right\}^{-1/2},$$

where

$$F(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}, \qquad \eta = \frac{e^2}{\hbar v},$$

$$h(\eta) = \mathrm{Re}\,\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} - \lg\eta,$$

$$R = \frac{\hbar^2}{Me^2} = 2.9\cdot10^{-12}\ \text{см}, \quad a = -7.7\cdot10^{-13}\ \text{см},$$

$$\gamma = 3.4\cdot10^{11}\ \text{MeV}^{-1}\ \text{см}^{-1}.$$

Substituting this expression into (3), we obtain

$$dw_P^{pp} = C_{pp}\frac{F(\eta)\,d\mathbf{P}}{F^2(\eta)E + \frac{\hbar^2}{M}\left[-\frac{1}{a} - \frac{h(\eta)}{R} + \gamma E\right]^2}. \tag{6}$$

As in the case of two neutrons, the spins of the protons are antiparallel, and the distribution does not depend on the angle of the vector $\mathbf{P}$.

7. Formula (3) is also applicable in the case when a high energy deuteron is incident on a nucleus. The distribution of the two nucleons according to energies of relative motion is given by the expression (4) if the charge of the nucleons does not change, and by expression (5) or (6) if there is a change in the charge of one of the nucleons.

It is possible to calculate the angular correlation of the emergent nucleons. For this, it is necessary to integrate the expression dw with respect to the component of the vector $\mathbf{P}$ parallel to the vector $\mathbf{P}_0$. ($\mathbf{P}_0$ is the momentum of the center of mass of the two nucleons in the laboratory coordinate system.)

Introducing the longitudinal ($\mathbf{P}_1$) and transverse ($\mathbf{P}_2$) components of the relative momentum of the nucleons, we obtain

$$d\mathbf{P} = dP_1\,dP_2\,2\pi P_2,$$

$$f(\vartheta)\,d\vartheta = 2\pi P_2\int w_P\,dP_1\frac{dP_2}{d\vartheta}\,d\vartheta.$$

Here θ is the angle between the exit directions of the nucleons. It is easy to see that when $P \ll P_0$

$$\vartheta = 4P_2/P_0,$$

hence the angular distribution is given by the expression

$$f(\vartheta)\,d\vartheta = \frac{2\pi P_0^2}{16}\int_{-\infty}^{+\infty} w_P\,dP_1\vartheta\,d\vartheta.$$

For neutron and proton we obtain from (4)

$$f(\vartheta)\,d\vartheta = \frac{P_0^2}{16}C'_{np}\int_{-\infty}^{+\infty}\left\{\frac{1}{[(P_1^2+P_2^2)/M]+\varepsilon_0} + \frac{a'}{[(P_1^2+P_2^2)/M]+\varepsilon_1}\right\}dP_1 2\pi\vartheta\,d\vartheta \tag{7}$$

$$= A_{np}\left\{\frac{1}{\sqrt{(4\varepsilon_0/E_0)+\vartheta^2}} + \frac{a'}{\sqrt{(4\varepsilon_1/E_0)+\vartheta^2}}\right\}\vartheta\,d\vartheta,$$

where E_0 is the energy of the center of mass of the two nucleons and is given by $E_0 = P_0^2/4M$. The quantity a' gives the fraction of the collisions which are accompanied by change of spin of the system neutron-proton in traversing the nucleus.

For two neutrons we obtain

$$f_{nn}(\vartheta)\,d\vartheta = A_{nn}\frac{\vartheta\,d\vartheta}{\sqrt{(4\varepsilon/E_0)+\vartheta^2}}. \tag{8}$$

In the case of two protons emerging after the collision Eq. (6) must be integrated with respect to the longitudinal momentum P_1. Upon numerical integration of Eq. (6), we obtain

$$f_{pp}(\vartheta)\,d\vartheta = A_{pp}\,\Phi(E_2)\,\vartheta\,d\vartheta; \tag{9}$$

here A_{pp} is a constant, and the "transverse" energy E_2 is related to the angle θ by the relation $E_2 = E_0\theta^2/4$, where E_0 is the energy of the incident deuteron. The function Φ which occurs in (9) may be presented in tabular form as follows:

E_2 (MeV)	0	1	2	3	4	5
Φ	1.00	0.82	0.59	0.40	0.34	0.31

[1] L.D. Landau and Ia. Smorodinskii, J. Exper. Theoret. Phys. **14**, 269 (1944)

[2] J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. **22**, 77 (1950)

Equations (3)-(9) hold under the condition that $\theta \ll \theta_0$ where θ_0 is the angle of deflection of the momentum of the center of mass in traversing the nucleus. (This condition is equivalent to the condition used in deriving formula (3): $P \ll \hbar q$).

8. In deriving the distributions according to the energy of the relative motion and the formulas of angular correlation, we used only the fact that the probability of the process (for large q) is proportional to the square of the function of two nucleons at small distances. It may be assumed, therefore, that formulas (4)-(9) will apply not only in the case when nucleons with small interaction energy have been formed as a result of a collision of a deuteron with a nucleon or nucleus, but also in other cases (e.g., collisions of α-particles with nuclei).

9. Expressions (5) and (8) which give energy and angular correlations of two slow neutrons formed as a result of a neutron-deuteron collision, may be used for the measurement of the very important quantity ϵ, which characterizes the interaction of two neutrons with antiparallel spins.

For this purpose it may be more convenient to study the energy of distribution of fast protons which result from an exchange collision. The energy distribution of the protons can be easily obtained from the distribution (5), if we restrict ourselves to the region of proton energy where the energy of relative motion of the two neutrons is sufficiently small. From the laws of conservation of energy and momentum one has

$$E_0 = E_p + \frac{P_p^2}{4M} + E_{nn} = \frac{3}{2}E_p + E_{nn}, \qquad (10)$$

where E_p and P_p are the energy and the momentum, respectively, of the proton, and E_{nn} is the energy of the two neutrons in their center of mass system. Let us denote the maximum energy of the protons by $E_p^m = 2/3\,E_0$; then

$$E_p = E_p^m - {}^2/_3\,E_{nn}. \qquad (11)$$

The energy distribution of the protons near E_p^m will be determined by the fact that the cross-section for the process is proportional to Eq. (5) and to the statistical weight of the final state,

$$f_p(E_p)\,dE_p = C_1\frac{1}{E_{nn}+\varepsilon}P^2\frac{dP}{dE_{nn}}dE_p = C\frac{\sqrt{E_{nn}}}{E_{nn}+\varepsilon}dE_p$$

Using (11), we obtain

$$f_p(E_p)\,dE_p = C\sqrt{{}^2/_3}\,[\sqrt{E_p^m - E_p}\,/\,(E_p^m - E_p + {}^2/_3\,\varepsilon)]\,dE_p. \qquad (12)$$

The distribution (12) has a maximum at $E_p^m - E_p = 2/3\,\epsilon$

If there exists a di-neutron, then along with distribution (12) there are monochromatic protons with energy given by $E_p' = E_p^m + 2/3\,\epsilon$ [as is seen from (10)].

The cross-section σ_0 for formation of the di-neutron can be connected quantitatively with the cross-section for formation of two neutrons in the free state. In fact, the ratio of the cross-sections of these two processes is given by the ratio $\phi_p(r_1)^2/\ \phi_0(r_1)^2$ where ϕ_0 is the function of the di-neutron.

The di-neutron function and the function ϕ_p must be expressed in the same form as for the system neutron-proton (replacing ϵ_0 by ϵ). The ratio of these expressions, as is easily seen from the theory of neutron-proton scattering and from deuteron theory, is given (for small r_1) by

$$\frac{|\varphi_p(r_1)|^2}{|\varphi_0(r_1)|^2} = \left(\frac{\hbar^2}{M}\right)^{3/2}\frac{2\pi}{\varepsilon^{1/2}}\,\frac{1}{E_{nn}+\varepsilon} \qquad (13)$$

with the functions ϕ_p normalized to unit volume and the function ϕ_0 normalized to unity. It is easily seen that with this normalization of the functions ϕ_p the ratio of the cross-sections will include the expression

$$\frac{3\pi}{(2\pi\hbar)^3}\,P^2\,\frac{dP}{dE_{nn}}\sqrt{\frac{E_p}{E_p'}}\,dE_p\,, \qquad (14)$$

which arises from the ratio of the statistical weights of the free and bound states.

Using (14) and (13), we obtain

$$d\sigma = \sigma_0\,\frac{3}{4\pi}\,\frac{\sqrt{E_{nn}}}{E_{nn}+\varepsilon}\sqrt{\frac{E_p}{E_p'\varepsilon}}\,dE_p$$

$$= \frac{\sigma_0}{2\pi}\sqrt{\frac{E_p}{E_p'\varepsilon'}}\,\frac{\sqrt{E_p^m - E_p}}{E_p^m - E_p + \varepsilon'}\,dE_p; \quad \varepsilon' = {}^2/_3\,\varepsilon. \qquad (15)$$

Expression (15) holds only for energies E_p which are close to E_p^m. For a rough estimate of the total cross-section for formation of free particles and the cross-section for formation of the bound state, let us assume that the expression (15) is valid in the entire range of energies E_p. Then, integrating (15) with respect to E_p from zero to E_p^m, we obtain

$$\sigma_0 / \int d\sigma \approx 4 \sqrt{\varepsilon' / E_p^m}. \qquad (16)$$

10. Formulas (10), (15) and (16) will also hold for the case of the energy distribution α-particles in the reaction:

$$H_3 + H_3 = n + n + \alpha.$$

The energy distribution of α-particles will have the same form as the distribution of protons in the reaction considered above: $n + H_2 = n + n + p$. The quantity ϵ' in the case of α-particles is $\epsilon' = 1/_3 \epsilon$. The chief difficulty in experiments of this kind is the necessity of resolving two maxima in the distribution curve (of the α-particles or protons). One maximum (at energy $E_p = E_p^m - 2/_3 \epsilon$ or $E_\alpha = E_\alpha^m - 1/_3 \epsilon$) corresponds to free neutrons while the other (at the energy $E_p = E_p^m + 2/_3 \epsilon$ or $E_\alpha = E_\alpha^m + 1/_3 \epsilon$) corresponds to the bound state of the two neutrons (if it exists).

Since the quantity ϵ is apparently very small (for two protons and for proton and neutron with antiparallel spins ϵ 100 keV), it follows that for a proof of the existence of a di-neutron an extremely precise determination of the energies of the α-particles (or protons) is necessary. This circumstance has not been taken into account in the experimental attempts to detect the di-neutron.

11. Expressions (15) and (16), obtained for the case of two neutrons, are, of course, valid also for the neutron-proton case, if by σ_0 one understands the cross-section for formation of the deuteron, and by $d\sigma$ the cross-section for formation of a free neutron and a free proton with parallel spins. It can be assumed that these expressions remain valid also for collisions of deuterons and of more complex particles with a nucleus. Equation (16) then gives an estimate of the ratio of the probability of formation of free nucleons to the probability of formation of the same nucleons in a bound state as a result of the nuclear reaction. The energy E_p^m is to be regarded as the maximum energy which is transferred to the nucleons in the given reaction. It is to be noted that an estimate of this ratio without taking into account of the nucleon interaction would be given by the ratio of the volume in the momentum space in the deuteron to that in the free state, i.e. it would have the form

$$\sigma_0 / \int d\sigma \sim (\varepsilon_0 / E^m)^{3/2}.$$

Taking into account the nucleon interaction, leads, as has been shown, to a much great probability of formation of the deuteron. Because of the resonance denominator in Eq. (4), one obtains the estimate given by (16). This formula explains the frequent appearance of deuterons in nuclear reactions.

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